June 2011

Fiscal Policy and Unemployment*

Abstract

This paper explores the interaction between fiscal policy and unemployment. It develops a dynamic economic model in which unemployment can arise but can be mitigated by tax cuts and public spending increases. Such policies are fiscally costly, but can be financed by issuing government debt. In the context of this model, the paper analyzes the simultaneous determination of fiscal policy and unemployment in long run equilibrium. Outcomes with both a benevolent government and political decision-making are studied. With political decision-making, the model yields an intuitively appealing positive theory of fiscal policy and unemployment.

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*For useful discussions we thank Roland Benabou, Karel Mertens, Thomas Sargent, and seminar participants at Yale University.

1 Introduction

An important role for fiscal policy is the mitigation of unemployment and stabilization of the economy.[1] Despite sceptism from some branches of the economics profession, politicians and policy-makers tend to be optimistic about the potential fiscal policy has in this regard. Around the world, countries facing downturns continue to pursue a variety of fiscal strategies, ranging from tax cuts to public works projects. Nonetheless, politicians' willingness to use fiscal policy to aggressively fight unemployment is tempered by high levels of debt. The main political barrier to deficit-financed tax cuts and public spending increases appears to be concern about the long-term burden of high debt.

This extensive practical experience with fiscal policy raises a number of basic positive public finance questions. In general, how do employment concerns impact the setting of taxes and public spending? When will government employ fiscal stimulus plans? What determines the size of these plans and how does this depend upon the economy's debt position? What will be the mix of tax cuts and public spending increases in stimulus plans and how will they be financed? What will be the overall effectiveness of fiscal policy in terms of reducing unemployment?

This paper presents a theory of the interaction between fiscal policy and unemployment that sheds light on these questions. The starting point for the theory is a simple dynamic economic model in which unemployment can arise but can be mitigated by tax cuts and public spending increases. Such policies are fiscally costly, but can be financed by issuing debt. This model is used to analyze the simultaneous determination of fiscal policy and unemployment in long run equilibrium. Outcomes with both a benevolent government and political decision-making are considered. With political decision-making, the model delivers an intuitively appealing positive theory of fiscal policy and unemployment.

The economic model has a public and private sector. The private sector consists of entrepreneurs who hire workers to produce a private good. The public sector hires workers to produce a public good. Public production is financed by a tax on the private sector. The government can also borrow and lend in the bond market. The private sector is affected by exogenous shocks (oil price hikes, for example) which impact entrepreneurs' demand for labor. Unemployment can arise because of a downwardly rigid wage. In the presence of unemployment, reducing taxes increases

[1] For an informative recent discussion of this role see Auerbach, Gale, and Harris (2010).

private sector hiring, while increasing public production creates public sector jobs. Thus, tax cuts and increases in public production reduce unemployment. However, both actions are costly for the government.

We show that in this model there would be no unemployment in the long run with a benevolent government. Moreover, the mix of public and private outputs would be optimal. The way in which the government achieves this first best outcome is by accumulating bond holdings. In the long run, in every period the government hires sufficient public sector workers to provide the Samuelson level of the public good and sets taxes so that the private sector has the incentive to hire the remaining workers. When the private sector is experiencing negative shocks, these taxes are sufficiently low that tax revenues fall short of the costs of public good provision. The earnings from government bond holdings are then used to finance this shortfall.

The benevolent government solution is provocative in showing how governments can use fiscal policy to completely circumvent the inefficiencies stemming from labor market frictions in the long run. The lesson suggested by the analysis is that no satisfactory theory of unemployment can abstract from how fiscal policy is chosen. Nonetheless, when interpreted as a positive theory, the solution is less interesting and this motivates considering political decision-making. To introduce this, we follow Battaglini and Coate (2007, 2008) in assuming that policy decisions are made in each period by a legislature consisting of representatives from different political districts. We also incorporate the friction that legislators can transfer revenues back to their districts.

With political decision-making, the government has no stock of bonds and, when the private sector experiences negative shocks, unemployment arises. Moreover, when these shocks occur, government mitigates unemployment with stimulus plans that are financed by increases in debt. These equilibrium stimulus plans typically involve both tax cuts and public production increases. When choosing such plans, the government balances the benefits of reducing unemployment with the costs of distorting the private-public output mix. In normal times, when the private sector is not experiencing negative shocks, the government reduces debt until it reaches a floor level. The existence of this floor level prevents bond accumulation as in the benevolent government solution. Even in normal times, the private-public output mix is distorted and unemployment can arise, depending on the economic and political fundamentals. With or without negative shocks, when there is unemployment, it will be higher the larger the government's debt level. High debt levels are therefore associated with high unemployment levels.

While there is a vast theoretical literature on fiscal policy, we are not aware of any work that systematically addresses the positive public finance questions that motivate this paper. Neoclassical theories of fiscal policy, such as the tax smoothing approach, assume frictionless labor markets and thus abstract from unemployment.[2] Traditional Keynesian models incorporate unemployment and allow consideration of the multiplier effects of changes in government spending and taxes. However, these models are static and do not incorporate debt and the costs of debt financing.[3] This limitation also applies to the literature in optimal taxation which has explored how optimal policies are chosen in the presence of involuntary unemployment.[4] The modern new Keynesian literature with its sophisticated dynamic general equilibrium models with sticky prices typically treats fiscal policy as exogenous.[5] Papers in this tradition that do focus on fiscal policy, analyze how government spending shocks impact the economy and quantify the possible magnitude of multiplier effects.[6]

Addressing the questions we are interested in requires a simple and tractable dynamic model. In creating such a model, we have made a number of strong assumptions. First, we employ a model without money and therefore abstract from monetary policy. This means that we cannot consider the important issue of whether the government would prefer to use monetary policy to achieve its policy objectives.[7] Second, we obtain unemployment by simply assuming a downwardly

[2] The tax smoothing approach, pioneered by Barro (1979), suggests that governments should use budget surpluses and deficits as a buffer to prevent tax rates from changing too sharply. Thus, governments will run deficits in recessions and surpluses in booms. Underlying the approach is the assumption that the deadweight costs of taxes are a convex function of the tax rate. Our model is not a tax smoothing model in the sense that, without the labor market frictions, taxation is not distortionary. Nonetheless, the downwardly rigid wage does create distortions and the economic role for debt is to smooth these across time. Moreover, our finding that, with a benevolent government, all distortions are eliminated in the long run parallels a similar finding for tax smoothing models (Aiyagari et al 2002). Furthermore, as in the model of this paper, this counter-factual long run prediction can be overcome by introducing political decision-making as shown by Battaglini and Coate (2008) and Barsheygan, Battaglini, and Coate (2010).

[3] For a nice exposition of the traditional Keynesian approach to fiscal policy see Peacock and Shaw (1971). Blinder and Solow (1973) discuss some of the complications associated with debt finance and extend the IS-LM model to try and capture some of these.

[4] This literature includes papers by Bovenberg and van der Ploeg (1996), Dreze (1985), Marchand, Pestieau, and Wibaut (1989), and Roberts (1982).

[5] See, for example, Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2003).

[6] See, for example, Christiano, Eichenbaum, and Rebelo (2009), Hall (2009), Mertens and Ravn (2010), and Woodford (2010).

[7] In an interesting recent contribution, Mankiw and Weinzierl (2011) study optimal fiscal and monetary policy in a two period general equilibrium model with sticky prices. Their analysis of fiscal policy differs from ours because they assume lump sum taxation so that Ricardian Equivalence holds.

rigid wage, as opposed to a more sophisticated micro-founded story.[8] This means that our analysis abstracts from any possible effects of fiscal policy on the underlying friction generating unemployment. Third, the source of cyclical fluctuations in our economy comes from the supply rather than the demand side. In our model, recessions arise because negative shocks to the private sector reduce the demand for labor. Labor market frictions prevent the wage from adjusting and the result is unemployment. This vision differs from the traditional and new Keynesian perspectives that emphasize the importance of shocks to consumer demand.[9]

While these strong assumptions undoubtedly represent limitations of our analysis, we nonetheless feel that our model provides an insightful framework in which to study activist fiscal policy. First, the model incorporates the two broad ways in which government can create jobs: *indirectly* by reducing taxes on the private sector, or *directly* through increasing public production. Second, the model allows consideration of two conceptually different types of activist fiscal policy: *balanced-budget policies* wherein tax cuts are financed by public spending decreases or visa versa, and *deficit-financed policies* wherein tax cuts and/or spending increases are financed by increases in public debt. Third, the mechanism by which taxes influence private sector employment in the model is consonant with arguments that are commonplace in the policy arena. For example, the main argument behind objections to eliminating the Bush tax cuts for those making $250,000 and above, was that it would lead small businesses to reduce their hiring during a time of high unemployment. Fourth, the mechanism by which high debt levels are costly for the economy also captures arguments that are commonly made by politicians and policy-makers. Higher debt levels imply larger debt service costs which require either greater taxes on the private sector and/or lower public spending. These policies, in turn, have negative consequences for jobs and the economy.

The organization of the remainder of the paper is as follows. Section 2 outlines the model. Section 3 studies fiscal policy and unemployment with a benevolent government. Section 4 introduces political decision-making, and Section 5 concludes.

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[8] There is a literature incorporating theories of unemployment into dynamic general equilibrium models (see Gali (1996) for a general discussion). Modelling options include matching and search frictions (Andolfatto 1996), union wage setting (Ardagna 2007), and efficiency wages (Burnside, Eichenbaum, and Fisher 1999).

[9] In the new Keynesian literature demand shocks are created by stochastic discount rates (see, for example, Christiano, Eichenbaum, and Rebelo (2009)). In Mankiw and Weinzerl's (2011) two period model, a demand shock in period one arises from a reduction in period two productivity which causes households to have lower expectations about period two income.

2 Model

The environment We consider an infinite horizon economy in which there are two final goods; a private good x and a public good g. There are two types of citizens; entrepreneurs and workers. Entrepreneurs produce the private good by combining labor l and an input z with their own effort. Workers are endowed with 1 unit of labor each period which they supply inelastically. The public good is produced by the government using labor.

There are n_e entrepreneurs and n_w workers where $n_e + n_w = 1$. Each entrepreneur produces with the Leontief production technology $x = A\min\{l, z, \epsilon\}$ where ϵ represents the entrepreneur's effort and A is a productivity parameter. The idea underlying this production technology is that when an entrepreneur hires more workers he must put in more effort to manage them. The public good production technology is $g = l$.

Workers' per period payoff function is $x + \gamma \ln g$, where γ measures the relative value of the public good. Entrepreneurs' per period payoff function is $x + \gamma \ln g - \xi\epsilon^2/2$ where the third term represents the disutility of providing entrepreneurial effort. All individuals discount the future at rate β.

There are markets for the private good, the input, and labor. The private good is the numeraire. The input is supplied by foreign suppliers and has an exogenous but variable price p_θ. We have in mind an input essential for production, such as energy. Each period, this price can take on one of two values p_L or p_H, where p_L is less than p_H and p_H is less than $A - \gamma/n_w$. We will say that the economy is in the *low cost* state when $\theta = L$ and the *high cost* state when $\theta = H$. The probability of the high cost state is α. The wage is denoted ω and the labor market operates under the constraint that the wage cannot go below an exogenous minimum $\underline{\omega}$. This friction is the source of unemployment. There is also a market for risk-free one period bonds. The assumption that citizens have quasi-linear utility implies that the equilibrium interest rate on these bonds is $\rho = 1/\beta - 1$.

To finance its activities, the government taxes entrepreneurs' incomes at rate τ. It can also borrow and lend in the bond market. Government debt is denoted by b and new borrowing by b'. The government is also able to distribute surplus revenues to citizens via lump sum transfers.

Market equilibrium At the beginning of each period, the cost state of the economy is revealed. The government repays existing debt and chooses the tax rate, public good, new borrowing, and transfers. It does this taking into account how its policies impact the market and the need to balance its budget.

To understand how policies impact the market, assume the cost state is θ, the tax rate is τ, and the public good level is g. Given a wage rate ω, each entrepreneur chooses hiring, the input, and effort, to maximize his utility

$$\max_{(l,z,\epsilon)} (1-\tau)(A\min\{l,z,\epsilon\} - p_\theta z - \omega l) - \xi\frac{\epsilon^2}{2}. \tag{1}$$

Obviously, the solution involves $z = \epsilon = l$. Substituting this into the objective function and maximizing with respect to l reveals that $l = (1-\tau)(A_\theta - \omega)/\xi$ where $A_\theta = A - p_\theta$. Aggregate labor demand from the private sector is therefore $n_e(1-\tau)(A_\theta - \omega)/\xi$. Labor demand from the public sector is g and labor supply is n_w. Setting demand equal to supply, the market clearing wage is

$$\omega = A_\theta - \xi(\frac{n_w - g}{n_e(1-\tau)}). \tag{2}$$

The minimum wage will bind if this wage is less than $\underline{\omega}$. In this case, the equilibrium wage is $\underline{\omega}$ and the unemployment rate is

$$u = \frac{n_w - g - n_e(1-\tau)(A_\theta - \underline{\omega})/\xi}{n_w}. \tag{3}$$

To sum up, in cost state θ with government policies τ and g, the equilibrium wage rate is

$$\omega_\theta = \begin{cases} \underline{\omega} & \text{if } A_\theta \leq \underline{\omega} + \xi(\frac{n_w-g}{n_e(1-\tau)}) \\ A_\theta - \xi(\frac{n_w-g}{n_e(1-\tau)}) & \text{if } A_\theta > \underline{\omega} + \xi(\frac{n_w-g}{n_e(1-\tau)}) \end{cases} \tag{4}$$

and the unemployment rate is

$$u_\theta = \begin{cases} \frac{n_w-g-n_e(1-\tau)(A_\theta-\underline{\omega})/\xi}{n_w} & \text{if } A_\theta \leq \underline{\omega} + \xi(\frac{n_w-g}{n_e(1-\tau)}) \\ 0 & \text{if } A_\theta > \underline{\omega} + \xi(\frac{n_w-g}{n_e(1-\tau)}). \end{cases} \tag{5}$$

When the minimum wage is binding, the unemployment rate is increasing in τ. Higher taxes cause entrepreneurs to put in less effort and this reduces private sector demand for workers. The unemployment rate is also decreasing in g because to produce more public goods, the government must hire more workers. When the minimum wage is not binding, the equilibrium wage is decreasing in τ and increasing in g.

Each entrepreneur earns profits of $\pi_\theta = (1-\tau)(A_\theta - \omega_\theta)^2/\xi$. Assuming he receives no government transfers and consumes his profits, an entrepreneur obtains a period payoff of

$$v_{e\theta}(\tau, g) = \frac{(A_\theta - \omega_\theta)^2(1-\tau)^2}{2\xi} + \gamma \ln g. \tag{6}$$

Jobs are randomly allocated among workers and so each worker obtains an expected period payoff

$$v_{w\theta}(\tau, g) = (1 - u_\theta)\omega_\theta + \gamma \ln g. \tag{7}$$

Again, this assumes that the worker receives no transfers and simply consumes his earnings.

Aggregate output of the private good is $x_\theta = n_e A(1-\tau)(A_\theta - \omega_\theta)/\xi$. Substituting in the expression for the equilibrium wage, we see that

$$x_\theta = \begin{cases} n_e A(1-\tau)(A_\theta - \underline{\omega})/\xi & \text{if } A_\theta \leq \underline{\omega} + \xi(\frac{n_w - g}{n_e(1-\tau)}) \\ A\,(n_w - g) & \text{if } A_\theta > \underline{\omega} + \xi(\frac{n_w - g}{n_e(1-\tau)}). \end{cases} \tag{8}$$

Observe that the tax rate has no impact on private sector output when the minimum wage constraint is not binding. This is because labor is inelastically supplied and as a consequence the wage adjusts to ensure full employment. A higher tax rate just leads to an offsetting reduction in the wage rate. However, when there is unemployment, tax hikes reduce private sector output because they lead entrepreneurs to reduce effort. Public good production has no effect on private output when there is unemployment, but reduces it when there is full employment.

The government budget constraint Having understood how markets respond to government policies, we can now formalize the government's budget constraint. Tax revenue is

$$R_\theta(\tau, \omega_\theta) = \tau(n_e \pi_\theta) = \tau n_e (1-\tau)(A_\theta - \omega_\theta)^2/\xi. \tag{9}$$

Total government revenue is therefore $R_\theta(\tau, \omega_\theta) + b'$. The cost of public good provision and debt repayment is $\omega_\theta g + b(1+\rho)$. The budget surplus available for transfers is the difference between $R_\theta(\tau, \omega_\theta) + b'$ and $\omega_\theta g + b(1+\rho)$. The government budget constraint is that this budget surplus be non-negative, which requires that

$$R_\theta(\tau, \omega_\theta) - \omega_\theta g \geq b(1+\rho) - b'. \tag{10}$$

There is also an upper limit $\overline{b}$ on the amount of debt the government can issue. This limit is motivated by the unwillingness of borrowers to hold bonds that they know will not be repaid.

If, in steady state, the government were borrowing an amount b such that the interest payments exceeded the maximum possible tax revenues in the high cost state; i.e., $\rho b > \max_{\tau} R_H(\tau, \underline{\omega})$, then, if the economy were in the high cost state, it would be unable to repay the debt *even if it provided no public goods or transfers.* The upper limit on debt is therefore $\overline{b} = \max_{\tau} R_H(\tau, \underline{\omega})/\rho$.

3 Benevolent government

It will prove instructive to break down the analysis of the benevolent government's solution into two parts. First, we study the static optimal policy problem for this economy. Thus, we ignore debt and, in the spirit of the optimal taxation literature, assume that the government faces an exogenous revenue requirement. Having understood how the static solution depends on the revenue requirement, we then introduce debt and study the dynamic policy choice problem. In the dynamic problem, the government's revenue requirement corresponds to the difference between debt repayment and new borrowing (as in (10)). Solving the dynamic model endogenizes the government's revenue requirement and completes the picture of the solution.

3.1 The static problem

The static optimal policy problem is to choose a tax rate τ and a level of public good g to maximize aggregate citizen utility subject to the requirement that revenues net of public production costs cover a revenue requirement r. To allow for the possibility of surpluses or deficits when debt is introduced, we assume that the revenue requirement can be positive or negative. Under the assumption that any surplus revenues are transferred to the citizens, this problem can be posed as:

$$\max_{(\tau,g)} \left\{ \begin{array}{c} R_\theta(\tau, \omega_\theta) - \omega_\theta g - r + n_e v_{e\theta}(\tau, g) + n_w v_{w\theta}(\tau, g) \\ s.t.\ R_\theta(\tau, \omega_\theta) - \omega_\theta g \geq r \end{array} \right\}. \tag{11}$$

What makes this problem non-standard is the endogenous wage and the possibility of unemployment. The problem of handling the endogenous wage can be simplified by noting that there is no loss of generality in assuming that the government always sets taxes sufficiently high so that the equilibrium wage equals $\underline{\omega}$. As noted earlier, taxes are non-distortionary when the wage exceeds $\underline{\omega}$ and the government has the ability to make transfers. Thus, if the wage exceeded $\underline{\omega}$, there would be no change in aggregate utility if the government raised taxes and simply redistributed the additional tax revenues back to the citizens. This observation allows us to write problem (11) as:

$$\max_{(\tau,g)} \left\{ \begin{array}{c} x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g - r \\ s.t.\ R_\theta(\tau, \underline{\omega}) - \underline{\omega} g \geq r \ \&\ g + \frac{x_\theta(\tau)}{A} \leq n_w \end{array} \right\}, \tag{12}$$

where $x_\theta(\tau)$ is the output of the private good when the tax rate is τ and the wage rate is $\underline{\omega}$ (see the top line of (8)).

Problem (12) has a simple interpretation. The objective function is the aggregate surplus generated by outputs $x_\theta(\tau)$ and g, less the revenue requirement.[10] The first inequality is the government budget constraint under the assumption that the wage is $\underline{\omega}$. The second inequality is the *resource constraint*: it requires that the demand for labor at wage $\underline{\omega}$ is less than or equal to the number of workers n_w.[11]

We will use a diagrammatic approach to characterize the solution to problem (12). Without loss of generality, we assume that r is less than or equal to the maximum possible tax revenue which is $\max_\tau R_\theta(\tau, \underline{\omega})$.[12] We also assume that unemployment would result if the government faced the maximal revenue requirement.[13] To understand our diagrammatic approach, consider Fig. 1.A. The tax rate is measured on the horizontal axis and the public good on the vertical. The upward sloping line is the resource constraint. Using the expression for $x_\theta(\tau)$ from (8), this line is described by

$$g = n_w - n_e(1-\tau)(A_\theta - \underline{\omega})/\xi. \tag{13}$$

At points along this line, there is full employment at the wage $\underline{\omega}$. Policies must be on or below this line and points below are associated with unemployment.

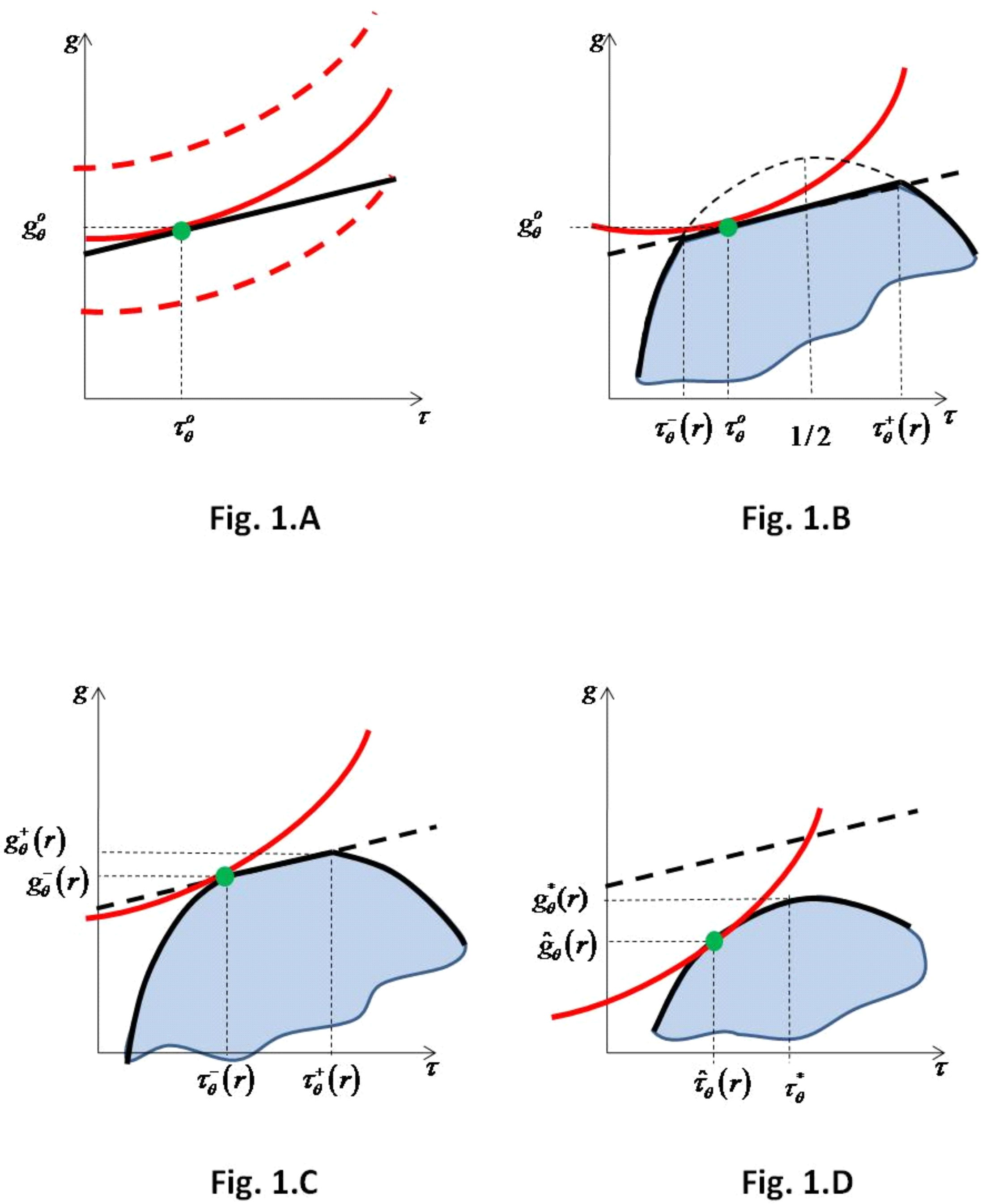
The upward sloping, convex curves represent the government's *indifference curves*. These curves tell us the government's preferences over different (τ, g) pairs. Indifference curves satisfy

[10] The expression for the surplus generated by $x_\theta(\tau)$ (the first two terms) reflects the fact that the surplus associated with the private good consists of the consumption benefits it generates less the costs associated with the input and entrepreneurial effort necessary to produce it.

[11] This constraint is required to ensure that the equilibrium wage is indeed $\underline{\omega}$.

[12] The revenue maximizing rate is $\tau = 1/2$ and thus the maximum revenue requirement is $n_e A_\theta(A_\theta - \underline{w})/4\xi$. Of course, if r were higher than this level, the problem would have no solution. In the dynamic model, however, this case will never arise.

[13] This assumption amounts to the requirement that n_w exceeds $n_e(A_\theta - \underline{\omega})/2\xi$. Assumption 1 below implies that this condition is satisfied for the high cost state.



g
g_θ^o
τ_θ^o
τ
Fig. 1.A
g
g_θ^o
$\tau_\theta^-(r)$ τ_θ^o $1/2$ $\tau_\theta^+(r)$
τ
Fig. 1.B
g
$g_\theta^+(r)$
$g_\theta^-(r)$
$\tau_\theta^-(r)$ $\tau_\theta^+(r)$
τ
Fig. 1.C
g
$g_\theta^*(r)$
$\hat{g}_\theta(r)$
$\hat{\tau}_\theta(r)$ τ_θ^*
τ
Fig. 1.D

Figure 1:

for some target utility level U

$$x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma\ln g = U. \tag{14}$$

Higher indifference curves are associated with higher utility levels, so utility is increasing as we move North-West. The indifference curves become flatter as we move South-East and the public good becomes more scarce.

The tangency point between the indifference curves and the full employment line defines the *first best* policies $(\tau_\theta^o, g_\theta^o)$. When these policies are in place, there is both full employment at wage $\underline{\omega}$ and the optimal mix of private and public outputs. It is straightforward to show that the optimal public good level in state θ is

$$g_\theta^o = \frac{\sqrt{(A_\theta n_e - \xi n_w)^2 + 4\xi n_e\gamma} - (A_\theta n_e - \xi n_w)}{2\xi}. \tag{15}$$

The associated tax rate τ_θ^o provides entrepreneurs with just the right incentive to employ those workers not employed in the public sector at the wage rate $\underline{\omega}$ and is given by

$$\tau_\theta^o = 1 - \frac{\xi\left(n_w - g_\theta^o\right)}{n_e(A_\theta - \underline{\omega})}. \tag{16}$$

In the remaining panels of Figure 1, we add the government's *budget line* - the locus of points that satisfy the budget constraint with equality. The budget line associated with revenue requirement r can be solved to yield

$$g = \frac{R_\theta(\tau, \underline{\omega})}{\underline{\omega}} - \frac{r}{\underline{\omega}}. \tag{17}$$

Policies must be on or below this line and points below are associated with positive transfers. Each budget line is hump shaped, with peak at $\tau = 1/2$. Increasing the revenue requirement shifts down the budget line but does not change the slope. Panels B, C, and D of Fig.1 represent increasing revenue requirements.

The feasible set of (τ, g) pairs for the optimal policy problem are those that lie below both the budget and resource constraints. This set is represented by the gray, cross hatched areas in Figure 1. Observe that the feasible set is (weakly) convex which makes the problem well-behaved.

We can now characterize the optimal policies. Three cases may be distinguished.

Case 1: Full Employment with No Distortions. In Fig. 1.B, the revenue requirement is small enough so that the first best point $(\tau_\theta^o, g_\theta^o)$ lies in the feasible set. The government can therefore select this and have revenue left over to rebate back to the citizens via a positive transfer. In this case, the budget constraint is not binding. This case arises when r is less than $r^o_\theta = R_\theta(\tau^o_\theta, \underline{\omega}) - \underline{\omega} g^o_\theta$.

Case 2: Full Employment with Distortions. In this case, the government distorts taxes and public production so as to achieve full employment. To be in this case, the budget line must lie above the resource constraint for some range of taxes. When it does, there will be a range of policies that can achieve full employment. In Fig. 1.C, for example, the government can achieve full employment by choosing any tax rate in the range $[\tau^-_\theta(r), \tau^+_\theta(r)]$ with associated level of public good given by (13). As r increases, this set shrinks both on the right and on the left (i.e., $\tau^+_\theta(r) - \tau^-_\theta(r) \to 0$).

If the government does choose full employment, it will choose the tax rate $\tau^-_\theta(r)$ with associated public good level $g^-_\theta(r)$ if τ^o_θ is to the left of $\tau^-_\theta(r)$. This is because $(\tau^-_\theta(r), g^-_\theta(r))$ is the closest point on the full employment line to the first best point $(\tau^o_\theta, g^o_\theta)$. By similar logic, if τ^o_θ is to the right of $\tau^+_\theta(r)$, the government will choose the tax rate $\tau^+_\theta(r)$ with associated public good level $g^+_\theta(r)$. The former case is illustrated in Fig. 1.C and Fig. 2.A, the latter in Fig. 2.B. In the former case, the government maintains full employment by *raising* taxes and public production above their first best levels. In the latter, it does so by *reducing* taxes and public production. In the former case, the government distorts the output mix *towards* public production and in the latter it distorts *away* from public production. These cases also generate different comparative static implications. In the former case, as the revenue requirement increases, taxes and public production increase, as illustrated in Fig. 2.A. In the latter, taxes and public production decrease as illustrated in Fig. 2.B.

It is notable that even in such a simple model, the direction of activist fiscal policy depends upon the underlying details of the economy. In the Appendix, we show that τ^o_θ is to the left of $\tau^-_\theta(r)$ if and only if:

$$\gamma < \frac{A_\theta}{2}\left(n_w - \frac{n_e A_\theta}{2\xi}\right). \tag{18}$$

Thus, this condition determines whether the government increases or reduces the size of government to achieve full employment. Intuitively, when γ is small, both taxes and public production are small in the unconstrained solution. This means that raising taxes will have a relatively small impact on private sector employment. Thus, when a higher revenue requirement must be met, a

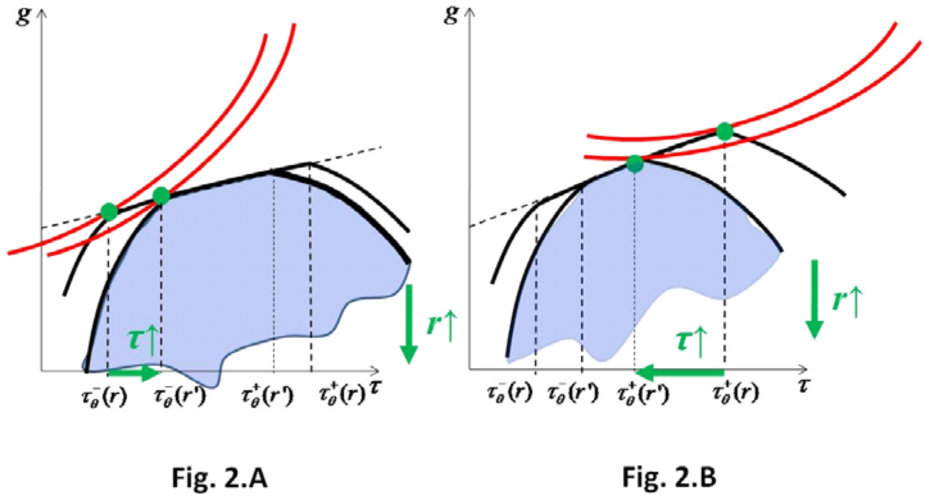


Figure 2:

fiscal surplus can be created by taxing the private sector and hiring the displaced workers in the public sector.

Case 3: Unemployment. In this case, the optimal policies involve unemployment. A sufficient condition for this case is that the budget line lies everywhere below the resource constraint as in Fig. 1.D. In this scenario, it is *impossible* for the government to maintain full employment. However, even when the government can maintain full employment, it will choose not to do so if the necessary distortions required in the output mix outweigh the benefits of full utilization of resources.

In fact, there will be a critical value of the revenue requirement below which the government will choose unemployment. Let $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$ denote the point at which the indifference curve is tangent to the budget line when the revenue requirement is r.[14] As r decreases, the tax rate decreases and the public production level increases, so $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$ moves to the North-West. Unemployment also decreases. Let r_θ^{**} denote the net borrowing level at which $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$ lies on the full employment line and let $(\tau_\theta^{**}, g_\theta^{**}) = (\widehat{\tau}_\theta(r_\theta^{**}), \widehat{g}_\theta(r_\theta^{**}))$. Then, when the revenue requirement is greater than r_θ^{**}, unemployment will result and the optimal policy will equal $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$. When

[14] The slope of the budget line is $n_e(1-2\tau)(A_\theta - \underline{\omega})^2/\xi\underline{\omega}$ and the slope of the indifference curves is $n_e(A_\theta - \underline{\omega})\left[\tau(A_\theta - \underline{\omega}) + \underline{\omega}\right] g/\xi\gamma$. Equating the two, we find that g must equal $\gamma(1-2\tau)(A_\theta - \underline{\omega})/\underline{\omega}\,(\tau(A_\theta - \underline{\omega}) + \underline{\omega})$. Combining this with (17) yields two equations which can be solved for $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$.

the revenue requirement is less than r_θ^{**}, the government will choose full employment.

The fact that the government will not necessarily choose to maintain full employment is illustrative of an important general lesson: the government will trade off minimizing unemployment with distorting the mix of public and private outputs. When full employment cannot be achieved, the unemployment minimizing policy involves the tax rate at which the slope of the budget line is equal to the slope of the full employment line, $\tau_\theta^* = (A_\theta - 2\underline{\omega})/2(A_\theta - \underline{\omega})$, with associated public good level $g_\theta^*(r)$ given by (17).[15] The optimal policy choice $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$ will not in general equal $(\tau_\theta^*, g_\theta^*(r))$, so that unemployment will be higher than it needs to be (as in Fig. 1.D). The optimal policy could involve a lower or higher tax rate than the unemployment minimizing rate depending upon the parameters of the economy and the size of the revenue requirement.[16] When it involves a lower tax rate, increasing the size of government would create jobs but the government holds back because the lost private output is more valuable than the additional public output. When the optimal policy involves a higher tax rate, reducing the size of government would create jobs but the government holds back for the opposite reason.

Putting all this together, we have the following characterization of the solution to problem (12).

Proposition 1 *The solution to problem (12) has the following properties.*

- *If $r \leq r_\theta^o$, the solution involves full employment with no distortions. The optimal policies are $(\tau_\theta^o, g_\theta^o)$ and are independent of the revenue requirement. In this range, an increase in r is absorbed by a reduction in government transfers.*

- *If $r \in (r_\theta^o, r_\theta^{**}]$, the solution involves full employment with distortions. If (18) is satisfied, the optimal policies are $(\tau_\theta^-(r), g_\theta^-(r))$ and the output mix is distorted in favor of the public good. In this range, as r increases both public production and the tax rate increase. If (18) is not satisfied, the optimal policies are $(\tau_\theta^+(r), g_\theta^+(r))$ and the output mix is distorted in favor of the private good. As r increases, both public production and the tax rate decrease.*

[15] The expression for τ_θ^* given in the text is derived in the proof of Proposition 1. This discussion assumes that $g_\theta^*(r) = \frac{R_\theta(\tau_\theta^*, \underline{w})}{\underline{w}} - \frac{r}{\underline{w}}$ is non-negative. If this is not the case, the unemployment minimizing tax rate is such that $R_\theta(\tau, \underline{w}) = r$ and the associated public good level is 0.

[16] As noted above, for sufficiently large r, the unemployment minimizing tax rate is such that $R_\theta(\tau, \underline{w}) = r$. However, the government will choose to provide some public good for any r less than the maximum level, implying that the tax rate exceeds the unemployment minimizing level.

- *If $r > r_\theta^{**}$, the solution involves unemployment. The optimal policies are $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$ and in general will differ from the unemployment minimizing policies. In this range, as r increases public production decreases, the tax rate increases, and unemployment increases.*

Proposition 1 tells us how taxes, public good production, and employment in each state depend on the government's revenue requirement.[17] A premise of the analysis is that the revenue requirement is exogenous. In a dynamic model, however, r is endogenous, depending on the amount of government debt that needs to be repaid and new borrowing.[18] Proposition 1, therefore, leaves a key question unanswered. In which of the three cases listed above should we expect the government to be in the long run?

3.2 Dynamics

With this understanding of optimal taxation and public production, we now bring debt into the picture. Debt will be helpful since it enables the government to smooth distortions across periods. The dynamic problem is to choose a time path of policies to maximize aggregate lifetime citizen utility. Since in equilibrium citizens are indifferent as to their allocation of consumption across time, their lifetime utility will equal the value of their initial bond holdings plus the payoff they would obtain if they simply consumed their net earnings and transfers in each period. Ignoring these initial bond holdings, the problem can therefore be formulated recursively as

$$V_\theta(b) = \max_{(\tau,g,b')} \left\{ \begin{array}{c} B_\theta(\tau, g, b', b, w_\theta) + n_e v_{e\theta}(\tau, g) + n_w v_{w\theta}(\tau, g) + \beta E V_{\theta'}(b') \\ s.t.\ B_\theta(\tau, g, b', b, w_\theta) \geq 0 \ \&\ b' \leq \overline{b} \end{array} \right\}, \tag{19}$$

where $V_\theta(b)$ is aggregate lifetime citizen utility in state θ with initial debt level b and $B_\theta(\cdot)$ denotes the budget surplus available for transfers.[19] Under this recursive formulation, in each period, given the cost state θ and initial debt level b, the government chooses the current tax rate τ, the public good level g, and new borrowing b'. Transfers are determined residually by $B_\theta(\tau, g, b', b, w_\theta)$.

[17] It is important to note that the solution described in Proposition 1 reflects our (w.l.o.g.) assumption that the government sets a tax rate such that the wage is $\underline{\omega}$. When r is less than r_θ^o, the government could equally well reduce the tax rate and let the wage rate rise above $\underline{\omega}$, compensating for the lost tax revenues by reducing transfers. Thus, in the case of full employment with no distortions, the optimal tax rate and level of transfers are not uniquely defined. In all the other cases, the solution must be exactly as described in Proposition 1.

[18] Specifically, from (10), we see that r will equal $(1+\rho)b - b'$.

[19] That is, $B_\theta(\tau, g, b', b, \omega_\theta) = R_\theta(\tau, \omega_\theta) + b' - \omega_\theta g - b(1+\rho)$.

As in the static problem, there is no loss of generality in assuming that the government always sets taxes sufficiently high that the wage is equal to $\underline{w}$. Thus, proceeding as in the static case and substituting $b(1+\rho)-b'$ for r, we may rewrite the government's problem as

$$V_\theta(b) = \max_{(\tau,g,b')} \left\{ \begin{array}{c} x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma\ln g + b' - b(1+\rho) + \beta EV_{\theta'}(b') \\ s.t.\ B_\theta(\tau,g,b',b,\underline{w}) \geq 0,\ g + \frac{x_\theta(\tau)}{A} \leq n_w \ \&\ b' \leq \overline{b} \end{array} \right\}. \tag{20}$$

We impose an assumption to focus the analysis on the natural case of interest. Specifically, we assume that when debt is zero the government is able to achieve the first best without borrowing in the low but not the high cost state. More precisely, we make:[20]

Assumption 1

$$R_H(\tau_H^o,\underline{w}) - \underline{w}g_H^o < 0 < R_L(\tau_L^o,\underline{w}) - \underline{w}g_L^o.$$

Recalling the definition of r_θ^o, the critical revenue requirement in Proposition 1 delineating Cases 1 and 2, Assumption 1 implies that r_H^o is negative and r_L^o is positive.

A solution to problem (20) consists of optimal policy functions $\{\tau_\theta(b), g_\theta(b), b'_\theta(b)\}$ for each state θ and value functions $V_H(b)$ and $V_L(b)$. By standard methods, it can be shown that there exists a solution and that the associated value functions are concave and differentiable. Corresponding to any solution, we can define $r_\theta(b) = (1+\rho)b - b'_\theta(b)$ to be the revenue requirement implied by the optimal policies in state θ with initial debt level b. Letting $(\tau_\theta^s(r), g_\theta^s(r))$ denote the optimal static policies described in Proposition 1, it is clear that $(\tau_\theta(b), g_\theta(b))$ will equal $(\tau_\theta^s(r_\theta(b)), g_\theta^s(r_\theta(b)))$. As discussed above, therefore, the key issue is to identify how the revenue requirement behaves in the long run. This will tell us which of the three cases described in Proposition 1 will arise.

To study the long run, note that given a solution to problem (20), for any initial debt level b_0 and sequence of shocks $\langle\theta_t\rangle$, we can associate a sequence of policies $\langle\tau_t, g_t, b'_t\rangle$.[21] The associated sequence of revenue requirements is then $\langle r_t\rangle$ where for all t, $r_t = (1+\rho)b'_{t-1} - b'_t$. The question is how these sequences behave as t becomes large. In fact, we can show that the probability that

[20] In terms of the fundamental parameters of the model this assumption amounts to:

$$n_eA_L - \xi n_w < n_e\left(\underline{w} + \frac{\gamma}{n_w}\right) - \frac{\xi\gamma}{\underline{w} + \frac{\gamma}{n_w}} < n_eA_H - \xi n_w.$$

[21] This sequence is defined inductively as follows: $(\tau_0, g_0, b'_0) = \left(\tau_{\theta_0}(b_0), g_{\theta_0}(b_0), b'_{\theta_0}(b_0)\right)$ and for all $t \geq 1$, $(\tau_t, g_t, b'_t) = \left(\tau_{\theta_t}(b'_{t-1}), g_{\theta_t}(b'_{t-1}), b'_{\theta_t}(b'_{t-1})\right)$.

r_t is less than or equal to r_H^o converges to one as t becomes large. From Proposition 1, we may conclude that, in the long run, the relevant case is the first. Thus we have:

Proposition 2 *In any solution to problem (20) the economy converges to full employment with no distortions.*

In the long run, therefore, in cost state θ, taxes and public production are $(\tau_\theta^o, g_\theta^o)$. In the high cost state $(\theta = H)$ public production is higher and tax revenues are lower. The increase in public production occurs because, while the benefit of public goods is state independent, the cost of the private good is higher. Lower tax revenues also reflect the fact that the private sector is less profitable.[22] Despite lower net tax revenues, the government is able to implement the first best policies in the high cost state in the long run because it has accumulated sufficient bond holdings.

Precisely how the government finances its activities is not tied down by the theory because there are multiple solutions to problem (20) and financing will depend on the details of the solution. The simplest solution is that the government gradually accumulates bonds until its debt level reaches r_H^o/ρ (recall that r_H^o is negative by Assumption 1). Once debt reaches this level, the steady state revenue requirement is r_H^o. This negative revenue requirement reflects the fact that the government is earning interest on its bond holdings. In the high cost state, these interest earnings are just sufficient to finance the shortfall in net tax revenues. In the low cost state, these interest earnings are rebated back to the citizens in a transfer along with the surplus net tax revenues r_L^o.[23] Intuitively, other solutions are possible because once debt has reached r_H^o/ρ, the government can further reduce it temporarily with no effect on citizens' utility.

Proposition 2 strikes us as interesting from a normative perspective. It suggests that, in the presence of labor market frictions, there is an intimate connection between fiscal policy and unemployment. In the model developed here, in the long run a benevolent government employs fiscal policy to circumvent the inefficiencies and achieve full employment with no distortions in all states. The general lesson hinted at is that no satisfactory theory of unemployment can abstract

[22] The impact on the tax rate of moving from the low cost to the high cost state is ambiguous. On the one hand, to hire any given number of workers, entrepreneurs need to be provided with lower taxes in the high cost state since workers are less profitable. On the other hand, entrepreneurs need to hire fewer workers because public production increases.

[23] As noted in footnote #17, in the low cost state, the government could equally well reduce the tax rate and let the wage rate rise above $\underline{w}$, compensating for the lost tax revenues by reducing transfers. In this case, we would observe wage reductions rather than transfer reductions when the economy moves from the low to the high cost state.

from how fiscal policy is chosen. Nonetheless, when interpreted as a positive theory of fiscal policy and unemployment, this benevolent government's solution is obviously unsatisfactory. This leads us to introduce political decision-making into the picture.

4 Political decision-making

We now study fiscal policy and unemployment with political decision-making. Our modeling strategy follows Battaglini and Coate (2007, 2008). Thus, we assume that the economy is divided into N identically sized political districts, each a microcosm of the economy as a whole. In each period, policy decisions are made by a legislature consisting of N representatives, one from each district. Each representative wishes to maximize the aggregate utility of the citizens in his district. In addition to choosing taxes, public goods, and borrowing, the legislature must also choose how to divide any budget surplus B_θ between the districts.

The decision-making process in the legislature follows a simple sequential protocol. At stage $j = 1, 2, ...$ of this process, a representative is randomly selected to make a proposal to the floor. A proposal consist of policies (τ, g, b') and district-specific transfers $(s_i)_{i=1}^N$ such that $B_\theta(\tau, g, b', b, \omega_\theta) \geq \sum_i s_i$ and $b' \leq \overline{b}$. If the proposal receives the votes of $Q < N$ representatives, then it is implemented and the legislature adjourns until the following period. If the proposal does not pass, then the process moves to stage $j+1$, and a representative is selected again to make a new proposal.[24]

Following the analysis in Battaglini and Coate (2008), it can be shown that in cost state θ with initial debt level b, the equilibrium policies $\{\tau_\theta(b), g_\theta(b), b'_\theta(b)\}$ are chosen to solve the maximization problem:

$$\max_{(\tau,g,b')} \left\{ \begin{array}{c} qB_\theta(\tau, g, b', b, \omega_\theta) + n_e v_{e\theta}(\tau, g) + n_w v_{w\theta}(\tau, g) + \beta E V_{\theta'}(b') \\ s.t.\ B_\theta(\tau, g, b', b, \omega_\theta) \geq 0 \ \&\ b' \leq \overline{b} \end{array} \right\}, \tag{21}$$

where $q = N/Q$ and $V_{\theta'}(b')$ is equilibrium aggregate lifetime citizen expected utility in state θ' with debt level b'. The equilibrium value functions $V_H(b)$ and $V_L(b)$ are defined recursively by:

$$V_\theta(b) = B_\theta(\tau_\theta(b), g_\theta(b), b'_\theta(b), b, \omega_\theta) + n_e v_{e\theta}(\tau_\theta(b), g_\theta(b)) + n_w v_{w\theta}(\tau_\theta(b), g_\theta(b)) + \beta E V_{\theta'}(b'_\theta(b)) \tag{22}$$

[24] This process may either continue indefinitely until a proposal is chosen, or may last for a finite number of stages as in Battaglini and Coate (2008): the analysis is basically the same. In Battaglini and Coate (2008) it is assumed that in the last stage, one representative is randomly picked to choose a policy; this representative is then required to choose a policy that divides the budget surplus evenly between districts.

for $\theta \in \{L, H\}$. Representatives' value functions, which reflect only aggregate utility in their respective districts, are given by $V_H(b)/N$ and $V_L(b)/N$.

Since it is not the focus of this work, we omit here the formal proof of this characterization of equilibrium.[25] The underlying intuition, however, is easily described. In the legislative bargaining process, the proposer chooses policies (τ, g, b') and transfers $(s_i)_{i=1}^N$ to maximize the expected welfare of his own district. The proposer may have to offer transfers to other districts to get his proposal approved. In this case, in a symmetric equilibrium, he offers the same transfer s to $Q-1$ other districts. At stage j of the process, the transfer s must be such that:

$$s + \frac{1}{N}\left[n_e v_{e\theta}\left(\tau, g\right) + n_w v_{w\theta}\left(\tau, g\right) + \beta E V_{\theta'}(b')\right] \geq \frac{V_\theta^{j+1}(b)}{N}, \tag{23}$$

where $V_\theta^{j+1}(b)$ is equilibrium aggregate lifetime citizen expected utility in state θ with initial debt level b at the beginning of stage $j+1$. A representative would indeed vote for the proposal only if the utility promised in this bargaining stage, the left hand side of (23), is at least as high as the utility of his outside option, $V_\theta^{j+1}(b)/N$.[26] In equilibrium, (23) must be satisfied with equality, implying that, at the margin, transfers depend on $\frac{1}{N}\left[n_e v_{e\theta}\left(\tau, g\right) + n_w v_{w\theta}\left(\tau, g\right) + \beta E V_{\theta'}(b')\right]$. This forces the proposer to internalize the opportunity cost of the policies (τ, g, b') for a fraction Q/N of the population. The resulting objective function for the proposer,

$$B_\theta(\tau, g, b', b, \omega_\theta) + \frac{Q}{N}\left[n_e v_{e\theta}\left(\tau, g\right) + n_w v_{w\theta}\left(\tau, g\right) + \beta E V_{\theta'}(b')\right], \tag{24}$$

is equivalent to the objective function in (21) (just multiply through by $q = N/Q$). The fact that representatives' value functions are described by (22) (divided by $1/N$) then follows from the fact that each representative is ex ante equally likely to be the proposer or, if not the proposer, to be included in the coalition whose districts receive transfers.

A convenient short-hand way of understanding the equilibrium is to imagine that in each period a minimum winning coalition (mwc) of Q representatives is randomly chosen and that this coalition collectively chooses policies to maximize its aggregate utility. Problem (21) reflects the coalition's maximization problem and, because membership in this coalition is random, all representatives are ex ante identical and have a common value function given by (22) (divided by

[25] See Battaglini and Coate (2008) for a more extensive discussion.

[26] The term $V_\theta^{j+1}(b)/N$ is endogenous in equilibrium, but from the point of view of the proposer it is given and so irrelevant for his policy choice.

$1/N$). In what follows, we will use this way of understanding the equilibrium and speak *as if* a randomly drawn mwc is choosing policy in each period.

For the purposes of the rest of this paper, we define a *political equilibrium* as consisting of policy functions $\{\tau_\theta(b), g_\theta(b), b'_\theta(b)\}$ for each state θ and value functions $V_H(b)$ and $V_L(b)$ such that: (i) the policy functions solve (21) given the value functions, and, (ii) the value functions satisfy (22) given the policy functions. An equilibrium is said to be *well-behaved* if the associated value functions $V_H(b)$ and $V_L(b)$ are concave for debt levels below $\overline{b}$. In the Appendix, we show:

Proposition 3 *There exists a well-behaved equilibrium.*

The equilibrium policies are characterized by solving problem (21). Again, there is no loss of generality in assuming that the mwc will always set taxes sufficiently high that the wage is $\underline{\omega}$. Indeed, because $q > 1$, it *must* be the case that the wage is $\underline{\omega}$, for the mwc would always raise taxes if it could extract more revenue with no deadweight cost. Thus, we can rewrite (21) as:

$$\max_{(\tau,g,b')} \left\{ \begin{array}{c} x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma\ln g + (q-1)\left(R_\theta(\tau,\underline{\omega}) - \underline{\omega}g\right) + q\left(b' - (1+\rho)b\right) + \beta EV_{\theta'}(b') \\ s.t.\ B_\theta(\tau,g,b',b,\underline{\omega}) \geq 0,\ g + \frac{x_\theta(\tau)}{A} \leq n_w \ \&\ b' \leq \overline{b} \end{array} \right\}. \tag{25}$$

To understand the equilibrium policies we follow the procedure used for the benevolent government case. First, we investigate the equilibrium tax and public good levels for a given revenue requirement. Then we understand the revenue requirements that arise in steady state by characterizing the equilibrium debt distribution.

4.1 The static problem

The equilibrium version of static problem (12) is

$$\max_{(\tau,g)} \left\{ \begin{array}{c} x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma\ln g + (q-1)\left(R_\theta(\tau,\underline{\omega}) - \underline{\omega}g\right) - qr \\ s.t.\ R_\theta(\tau,\underline{\omega}) - \underline{\omega}g \geq r \ \&\ g + \frac{x_\theta(\tau)}{A} \leq n_w \end{array} \right\}. \tag{26}$$

The key difference between this and problem (12) is that, since $q > 1$, the mwc puts more weight on tax revenues net of public production costs, $R_\theta(\tau,\underline{\omega}) - \underline{\omega}g$, than does a benevolent government. This implies that the indifference curves associated with the objective function (26) are steeper in (τ, g) space.

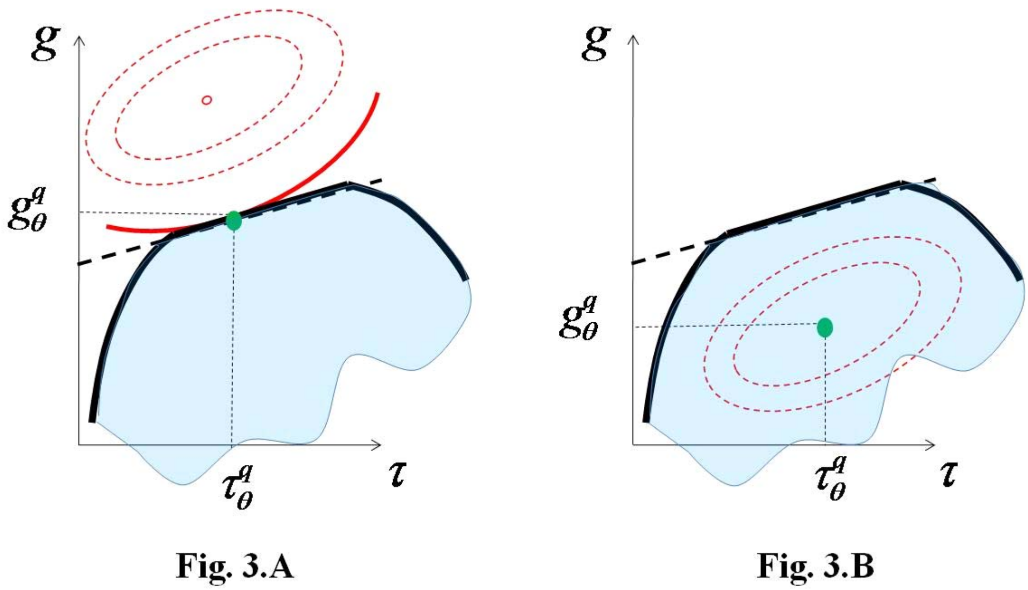


Figure 3:

There are two possible cases to consider, depending on the size of q. This variable, which measures (inversely) the fraction of the legislature required to form a mwc, turns out to play a key role in determining the nature of the solution.

4.1.1 Low q

Following the strategy used to study the static problem, first consider what happens when the revenue requirement is so low that the budget constraint is not binding. In the case represented in Fig. 3.A, the mwc's optimal choice $(\tau_\theta^q, g_\theta^q)$ corresponds to the point of tangency between the indifference curve and the resource constraint. This can be solved to yield:

$$(\tau_\theta^q, g_\theta^q) = \left(1 - \frac{\xi\,(n_w - g_\theta^q)}{n_e(A_\theta - \underline{\omega})}, \frac{\sqrt{(qA_\theta n_e - (2q-1)\xi n_w)^2 + (2q-1)4\xi n_e \gamma} - (qA_\theta n_e - (2q-1)\xi n_w)}{(2q-1)2\xi}\right). \tag{27}$$

This case is similar to the situation illustrated in Fig. 1.B, although now the indifference curves are steeper, so g_θ^q is smaller than g_θ^o.

Define r_θ^q to be the revenue requirement equal to $R_\theta(\tau_\theta^q, \underline{\omega}) - \underline{\omega} g_\theta^q$. Then, if the revenue requirement is less than or equal to r_θ^q the mwc will choose the optimal tax and public production levels (27). Even though we have full employment and distortions in this case, the reason for the distortions is not to create jobs but to extract revenue for transfers. Thus, we refer to this case as *full employment with optimal revenue extraction.*

If r exceeds r_θ^q the mwc will not make transfers to their districts and the budget constraint will bind. The optimal policy will then solve:

$$\max_{(\tau,g)} \left\{ \begin{array}{c} x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g - qr \\ s.t.\ R_\theta(\tau, \underline{\omega}) - \underline{\omega} g \geq r \ \&\ g + \frac{x_\theta(\tau)}{A} \leq n_w \end{array} \right\}. \tag{28}$$

This is equivalent to the problem studied in Section 3 and thus the solution will be as described by Proposition 1.[27] There will be two regions, one with full employment with distortions ($r \leq r_\theta^{**}$) and one with unemployment ($r > r_\theta^{**}$).

The case represented in Fig. 3.A arises when the optimal policy ignoring the budget constraint is at the tangency point with the resource constraint. The other possibility, illustrated in Fig. 3.B, is that the optimal policy lies in the interior of the feasible set and the resource constraint is not binding. We show in the Appendix that the resource constraint binds if and only if q is less than q_θ^*, where q_θ^* is defined by:

$$n_e \left[\frac{q(A_\theta - \underline{\omega}) + \underline{\omega}}{\xi(2q-1)}\right] + \frac{\gamma}{(q-1)\,\underline{\omega}} = n_w. \tag{29}$$

The analogy to Proposition 1 in this case is therefore:

Proposition 4 *When $q < q_\theta^*$ the solution to problem (26) has the following properties.*

- *If $r \leq r_\theta^q$, the solution involves full employment with optimal revenue extraction. The equilibrium policies are $(\tau_\theta^q, g_\theta^q)$ and are independent of the revenue requirement.*
- *If $r \in (r_\theta^q, r_\theta^{**}]$, the solution involves full employment with distortions. If (18) is satisfied, the equilibrium policies are $(\tau_\theta^-(r), g_\theta^-(r))$. In this range, as r increases both public production and the tax rate increase. If (18) is not satisfied, the optimal policies are $(\tau_\theta^+(r), g_\theta^+(r))$. As r increases, both public production and the tax rate decrease.*
- *If $r > r_\theta^{**}$, the solution involves unemployment. The equilibrium policies are $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$. In this range, as r increases public production decreases, the tax rate increases, and unemployment increases.*

[27] The fact that r is multiplied by q in the objective function has no effect on the optimal policies, since qr is just a constant.

4.1.2 High q

When q exceeds q_θ^* and the revenue requirement is so low that the budget constraint does not bind, we are in the case represented in Fig. 3.B. Intuitively, the mwc wants to keep taxes high and public production low to extract revenue to finance transfers to their districts. When q exceeds q_θ^*, taxes are sufficiently high and public production sufficiently low, that unemployment arises. Optimal extractive policies in this case are:

$$(\tau_\theta^q, g_\theta^q) = \left(\frac{(q-1)\, A_\theta - q\underline{\omega}}{(A_\theta - \underline{\omega})\,(2q-1)}, \frac{\gamma}{(q-1)\,\underline{\omega}} \right). \tag{30}$$

As before, define r_θ^q to be the revenue requirement equal to $R_\theta(\tau_\theta^q, \underline{\omega}) - \underline{\omega} g_\theta^q$. Then, if r is less than or equal to r_θ^q the mwc will choose (30). As in the low q case, policies in this case are driven by the desire to extract revenue rather than to mitigate unemployment. We therefore refer to this case as *unemployment with optimal revenue extraction.*

If r exceeds r_θ^q the mwc will not make transfers and the optimal policies will be as described in Proposition 1. Since there is unemployment at $(\tau_\theta^q, g_\theta^q)$, it must be the case that r_θ^q exceeds r_θ^{**} and so we will have unemployment. To summarize:

Proposition 5 *When $q > q_\theta^*$ the solution to problem (26) has the following properties.*

- *If $r \leq r_\theta^q$, the solution involves unemployment with optimal revenue extraction. The equilibrium policies are $(\tau_\theta^q, g_\theta^q)$ and are independent of the revenue requirement.*

- *If $r > r_\theta^q$, the solution involves unemployment. The equilibrium policies are $(\widehat{\tau}_\theta(r), \widehat{g}_\theta(r))$. In this range, as r increases public production decreases, the tax rate increases, and unemployment increases.*

4.2 Dynamics

As for the benevolent government case, define $r_\theta(b) = (1+\rho)b - b_\theta'(b)$ to be the revenue requirement implied by the equilibrium policies in state θ with initial debt level b. Letting $(\tau_\theta^e(r), g_\theta^e(r))$ denote the static equilibrium policies described in Propositions 4 and 5, it is clear that $(\tau_\theta(b), g_\theta(b))$ will equal $(\tau_\theta^e(r_\theta(b)), g_\theta^e(r_\theta(b)))$. As in the previous section, therefore, the key issue is to identify which of the cases described in Propositions 4 and 5 will arise in the long run. This requires understanding the long run behavior of debt.

Given the equilibrium policy functions, for any initial debt level b, we can define $H(b, b')$ to be the probability that next period's debt level will be less than b'. Given a distribution $\psi_{t-1}(b)$ of debt at time $t-1$, the distribution at time t is $\psi_t(b') = \int_b H(b, b')d\psi_{t-1}(b)$. A distribution $\psi^*(b')$ is said to be an *invariant distribution* if

$$\psi^*(b') = \int_b H(b, b')d\psi^*(b).$$

If it exists, the invariant distribution describes the steady state of the government's debt distribution. We now have:

Proposition 6 *There exists a debt level $b^q \in (r_H^o/\rho, \overline{b})$ such that the equilibrium debt distribution converges to a unique, non-degenerate, invariant distribution with full support on $[b^q, \overline{b})$. The dynamic pattern of debt is counter-cyclical: the government expands debt when private sector costs are high and contracts debt when costs are low until it reaches the floor level b^q.*

The floor debt level b^q described in Proposition 6 prevents the government from accumulating a sufficiently large buffer stock of assets that there is no need to issue new debt. Once the debt level has reached b^q, the mwc prefers to divert surplus revenues to transfers rather than to paying down more debt. This is analogous to the results of Battaglini and Coate (2008) and Barshegyan, Battaglini, and Coate (2010) for the tax smoothing model. The debt level b^q depends on the fundamentals of the economy and can be characterized following the analysis in Battaglini and Coate (2008), but these details are not central to our mission here and so we relegate them to the Appendix. For now, we will simply assume that b^q is positive, which seems the empirically relevant case.

In terms of revenue requirements, Proposition 6 implies that in steady state $r_H(b)$ is less than ρb and $r_L(b)$ is at least as big as ρb with the inequality holding strictly for b larger than b^q. Two further important properties are established in the Appendix. First, for each state θ, $r_\theta(b)$ is increasing in b, so that higher debt levels result in greater fiscal pressure on the government. Second, the floor debt level b^q is such that $r_H(b^q)$ exceeds r_H^q while $r_L(b^q)$ is less than r_L^q. This implies that the economy never reaches the first cases of Propositions 4 and 5 in the high cost state, but will do so in the low cost state for sufficiently low debt levels. Letting b_L^q be the debt level such that $r_L(b_L^q)$ is equal to r_L^q, this set of "sufficiently low debt levels" is $[b^q, b_L^q]$.

Turning to employment levels, from Proposition 6, we know that when q exceeds q_θ^*, we will necessarily have unemployment in state θ whatever revenue requirement the government faces.

From Proposition 5, when q is less than q^*_θ, we will have unemployment in state θ whenever the revenue requirement exceeds r^{**}_θ. This will occur with positive probability if $r_\theta(\overline{b})$ exceeds r^{**}_θ. Since $r_H(\overline{b})$ is equal to $\max_\tau R_H(\tau, \underline{\omega})$, a sufficient condition for unemployment to arise in the high cost state is that private sector labor demand is less than the number of workers when taxes are revenue maximizing. Formally, this amounts to the condition that n_w exceeds $n_e\left(A_H - \underline{\omega}\right)/2\xi$, which is implied by Assumption 1. Assumption 1 does not guarantee that $r_L(\overline{b})$ exceeds r^{**}_L, and so it is possible that unemployment does not occur in the low cost state when q is less than q^*_L. For our next result, it is convenient to define b^{**}_θ to be the maximal debt level in the set $\left[b^q, \overline{b}\right]$ such that that $r_\theta(b)$ is lower than or equal to r^{**}_θ. In the high cost state, b^{**}_H is less than $\overline{b}$, while in the low cost state it could be that b^{**}_L equals $\overline{b}$.

Proposition 7 *The equilibrium debt distribution has the following implications for employment levels and policies.*

- *If $q > q^*_L$, there is unemployment in the high cost state. In the low cost state, there is unemployment with optimal revenue extraction if $b \in [b^q, b^q_L]$ and unemployment if $b > b^q_L$. Unemployment is increasing in b in the high cost state and in the low cost state when $b > b^q_L$. For given b, unemployment is greater in the high than the low cost state.*

- *If $q \in (q^*_H, q^*_L)$, there is unemployment in the high cost state. In the low cost state, there is full employment with optimal revenue extraction if $b \in [b^q, b^q_L]$, full employment with distortions if $[b^q_L, b^{**}_L]$, and unemployment if $b > b^{**}_L$. Unemployment is increasing in b in the high cost state and in the low cost state when $b > b^{**}_L$. For given b, unemployment is greater in the high than the low cost state.*

- *If q is less than q^*_H, in the high cost state, there is full employment with distortions if $[b^q, b^{**}_H]$ and unemployment if $b > b^{**}_H$. In the low cost state, there is full employment with optimal revenue extraction if $b \in [b^q, b^q_L]$, full employment with distortions if $[b^q_L, b^{**}_L]$, and unemployment if $b > b^{**}_L$. Unemployment is increasing in b in the high cost state when $b > b^{**}_H$ and in the low cost state when $b > b^{**}_L$. For given b, unemployment is greater in the high than the low cost state when $b > b^{**}_H$.*

Proposition 7 shows that the model is consistent with a variety of possible employment patterns ranging from unemployment in both high and low cost states to full employment in both states except at high debt levels. There are two general lessons from the Proposition. First, for any given debt level in the support of the equilibrium distribution, employment levels are weakly higher in the low cost state and strictly higher for debt levels above some critical level. What this means is that negative shocks to the private sector translate into higher unemployment levels. Second, employment levels in both states are weakly decreasing in the economy's debt level and strictly so for debt levels above a critical level. Thus, higher debt leads to greater unemployment.

To illustrate the workings of the model, consider the case in which q is between q_L^* and q_H^*. In this case, there is always unemployment in high cost states but full employment in low cost states for suitably low debt levels. The government mitigates unemployment in high cost states by issuing debt. If b is less than b_L^{**}, then a return to the low cost state will be sufficient to restore full employment. If the economy is in the high cost state for a sufficiently long period of time, however, debt will increase beyond this level and unemployment will persist even when the low cost state returns. Eventually, however, the economy returns to full employment. When the low cost state returns, the legislators reduce debt. If the low cost state persists, debt will fall below b_L^{**}. At this point, full employment is achieved with activist fiscal policies. When debt falls below b_L^q, we achieve the optimal policies for the mwc and transfers will be made. Employment concerns no longer distort policies.

4.3 Equilibrium stimulus plans

In the steady state of the political equilibrium, when private sector costs are high, the government expands debt and the funds are used to mitigate unemployment.[28] The government therefore employs fiscal stimulus plans, as conventionally defined. By studying the size of these stimulus plans and the changes in policy they finance, we can obtain a positive theory of fiscal stimulus. More specifically, in the high cost state, we can interpret $\rho b - r_H(b)$ as the magnitude of the stimulus, since this measures the amount of additional resources obtained by the government to finance fiscal policy changes. An understanding of how the stimulus funds are used can be obtained by comparing the equilibrium tax and public good policies with the policies that would be optimal if the debt level were held constant.

The simplest case to consider is when the stimulus package does not completely eliminate

[28] Even when q is less than q_H^* and b is less than b_H^{**}, Assumption 1 implies that there will be unemployment prior to government stimulus if b^q is positive.

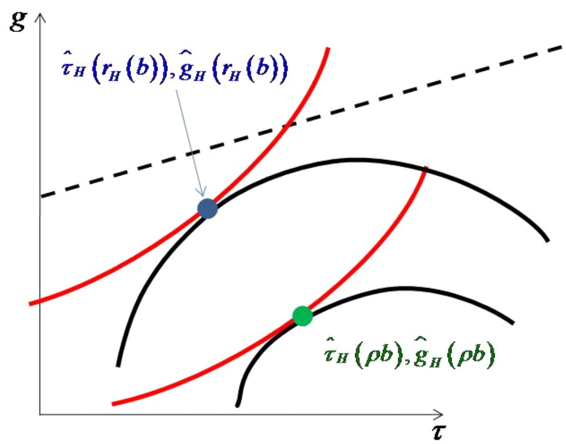


Figure 4:

unemployment. This must be the case when q exceeds q_H^*. Moreover, even when this is not the case, unemployment will remain post-stimulus whenever the initial debt level exceeds b_H^{**}. Drawing on the analysis in Section 3, Fig. 4 illustrates what happens in this case. From Propositions 4 and 5, the policies that would be chosen if the debt level were held constant are $(\widehat{\tau}_H(\rho b), \widehat{g}_H(\rho b))$. The reduction in the revenue requirement made possible by the stimulus funds, shifts the budget line up and permits a new policy choice $(\widehat{\tau}_H(r_H(b)), \widehat{g}_H(r_H(b)))$. As discussed in Section 3, in the unemployment range, the tax rate is increasing in the revenue requirement and public production is decreasing. Thus, we know that $\widehat{\tau}_H(r_H(b))$ is less than $\widehat{\tau}_H(\rho b)$ and that $\widehat{g}_H(r_H(b))$ exceeds $\widehat{g}_H(\rho b)$, implying that stimulus funds will be used for both tax cuts and increases in public production.

In terms of effectiveness, we know from Section 3 that if $\widehat{\tau}_H(r_H(b))$ is less than τ_H^* (the tax rate at which the slope of the budget line equals that of the resource constraint) then reducing the tax cut slightly and using the revenues to finance a slightly larger public production increase will produce a bigger reduction in unemployment. Conversely, if $\widehat{\tau}_H(r_H(b))$ exceeds τ_H^* then reducing the public production increase and using the revenues to finance a slightly larger tax cut will produce a bigger reduction in unemployment. That the government chooses a tax rate different than τ_H^* reflects the fact that it cares not only about unemployment but also the mix of public and private outputs. When $\widehat{\tau}_H(r_H(b))$ is less than τ_H^* it prefers a smaller level of public production than that associated with the tax rate τ_H^*, while if $\widehat{\tau}_H(r_H(b))$ exceeds τ_H^* it prefers a larger level of public production. Both situations are possible depending on the parameters.

Whether $\widehat{\tau}_H(r_H(b))$ is less or greater than τ_H^* will depend partially on the initial debt level. As b approaches $\overline{b}$, the government will always choose too small a tax cut. This is irrespective of its preferences as measured by γ. This reflects that as b approaches $\overline{b}$, the equilibrium tax rate approaches $1/2$ which exceeds τ_H^*.

One way of thinking about these results concerning the comparison of $\widehat{\tau}_H(r_H(b))$ and τ_H^* is in terms of multipliers. It is commonplace in the empirical literature to try to evaluate the multipliers associated with different stimulus measures.[29] The multiplier associated with a particular stimulus measure is defined to be the change in GDP divided by the budgetary cost of the measure. In our model, GDP equals private sector output plus the cost of public production. When there is unemployment, the public production multiplier is 1 and the tax cut multiplier is approximately $A_H/\left(1-2\widehat{\tau}_H(r_H(b))\right)(A_H-\underline{w})$. The tax cut multiplier will exceed the public production multiplier if $\widehat{\tau}_H(r_H(b))$ exceeds τ_H^* and be less than the public production multiplier if $\widehat{\tau}_H(r_H(b))$ is less than τ_H^*. The analysis illustrates why we should not expect the government to choose policies in such a way as to equate multipliers across instruments.[30] Tax cuts and public production increases have different implications for the mix of public and private outputs. A further important point to note is that the tax multiplier is highly non-linear. Tax cuts will be more effective the larger is the tax rate and the tax rate reflects the economy's initial debt level.

When the stimulus package eliminates unemployment, as would be the case when q exceeds q_H^* and the initial debt level is less than b_H^{**}, matters are more complicated. This is because of the non-monotonic behavior of policies in the full employment with distortions range identified in Proposition 1. In particular, we will not necessarily get both tax cuts and an increase in public production. Fig. 5.A illustrates a case in which the stimulus package involves not only using all the stimulus funds to fund tax cuts but also reducing public production to supplement the stimulus funds. Fig. 5.B illustrates a case in which the stimulus package involves increases in both taxes and public production. The model is therefore consistent with a variety of possible stimulus plans.

It is interesting to understand how the magnitude of the stimulus as measured by $\rho b - r_H(b)$

[29] Papers trying to measure the multiplier impacts of different policies include Alesina and Ardagna (2010), Barro and Redlick (2011), Blanchard and Perotti (2002), Mountford and Uhlig (2009), Ramey (2011), Romer and Romer (2010), Serrato and Wingender (2011), and Shoag (2010). A central issue in this literature is the relative size of tax cut and public spending multipliers.

[30] This point is also made by Mankiw and Weinzierl (2011).

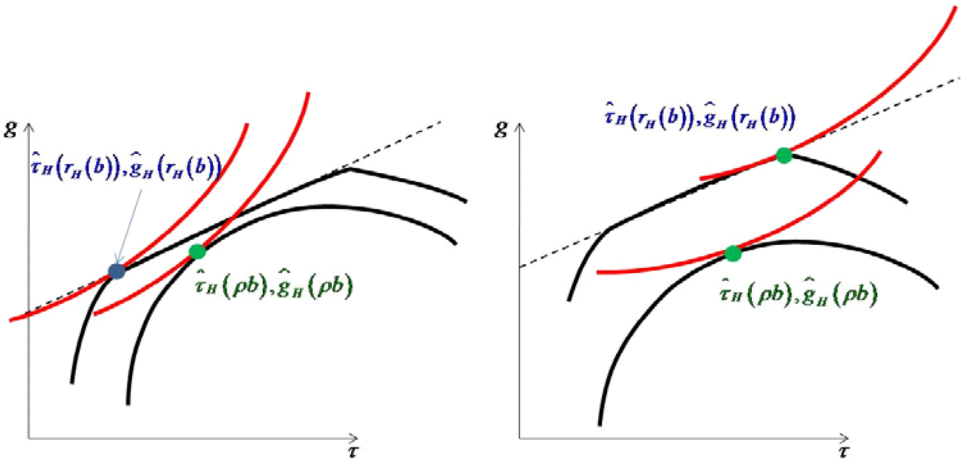


Figure 5:

depends on the initial debt level b. Note first that as b approaches its maximum level $\overline{b}$, the size of the stimulus must converge to zero. Interpreting the distance $\overline{b} - b$ as the economy's *fiscal space*, this result is simply saying that when the economy's fiscal space becomes very small (as a result, say, of a sequence of negative shocks or weak political institutions), its efforts to fight further negative shocks with fiscal policy will necessarily be limited.[31] It is tempting to conclude more generally, that the size of the stimulus as measured by $\rho b - r_H(b)$ should depend negatively on the initial debt level b. While we conjecture that this will typically be the case, it is not something that we have been able to show analytically. It should also be noted that even if this were the case, the effectiveness of stimulus plans will not necessarily be decreasing in the economy's fiscal space. This is because as the economy's fiscal space contracts, taxes on the private sector increase to finance debt repayment. When taxes are high, the tax multiplier is high, meaning that small tax cuts can create large gains in employment.

5 Conclusion

This paper has developed a simple dynamic model in which to explore the interaction between fiscal policy and unemployment. Two distinct scenarios have been considered, one in which policies are

[31] For more on the concept of fiscal space and an attempt to measure it see Ostroy, Ghosh, Kim and Qureshi (2010).

chosen by a benevolent government and the other with political decision-making. The benevolent government solution offers interesting, but clearly counter-factual predictions. The equilibrium with political decision-making offers an intuitively appealing account of the behavior of fiscal policy and unemployment.

With political decision-making, unemployment will arise when the private sector experiences negative shocks. To mitigate this unemployment, the government will employ debt-financed fiscal stimulus plans, which will generally involve both tax cuts and public production increases. In normal times, the government will contract debt until it reaches a floor level. Depending on the extent of political frictions, unemployment can arise even in normal times. At all times, unemployment levels are increasing in the economy's debt level. Except when debt is at its floor level, the mix of public and private output is distorted by employment concerns. The direction of distortion in terms of whether the size of government is too large or too small, depends upon the underlying parameters of the economy and the economy's debt level.

There are a number of different directions in which the theoretical model developed here might usefully be extended. First, it would be interesting to introduce class conflict into the political decision-making. The current model limits the conflict among citizens to disagreements concerning the allocation of transfers between districts. This is made possible by assuming that each legislator behaves so as to maximize the aggregate utility of the citizens in his district. Alternatively, we could assume that legislators either represent workers or entrepreneurs in their districts. This would introduce an additional conflict over policies in the sense that workers prefer policies that keep wages and employment high, while entrepreneurs prefer policies which keep profits high. Such class conflict may have important implications for the choice of fiscal policy. Second, it would be interesting to introduce capital into entrepreneurs' production technologies and model their choice of capital investment. In this environment, debt-financed stimulus plans would presumably lead entrepreneurs to cut back capital investment as they anticipate higher future taxes. This might both limit the effectiveness and raise the costs of using fiscal stimulus.

It would also be interesting to explore the empirical implications of the theory. There is an extensive literature on the cyclical behavior of fiscal policy.[32] The benchmark model for this literature has been the tax smoothing model. The model presented in this paper offers an alternative

[32] See, for example, Alesina, Campante, and Tabellini (2008), Barro (1986), Barshegyan, Battaglini, and Coate (2010), Furceri and Karras (2011), Gavin and Perotti (1997), Lane (2003), and Talvi and Vegh (2005).

benchmark for the behavior of fiscal policy over the cycle. It would be interesting to explore how well its qualitative predictions concerning the correlation of policy variables and output match the data. The novel feature of this model for this literature is that it offers predictions concerning the behavior of unemployment and its relationship with debt. These predictions should be explored. At a less aggregative level, the model also offers predictions concerning the size and design of stimulus plans which might usefully be investigated.

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6 Appendix

6.1 Proof of Proposition 1

The Lagrangian for Problem (12) is

$$L = x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma\ln g - r + \lambda\left(R_\theta(\tau,\underline{\omega}) - \underline{\omega}g - r\right) + \mu\left(n_w - g - \frac{x_\theta(\tau)}{A}\right).$$

Thus, λ is the multiplier on the budget constraint and μ is the multiplier on the resource constraint. Using the expressions for $x_\theta(\tau)$ and $R_\theta(\tau,\underline{\omega})$ in (8) and (9), the first order conditions with respect to g and τ are

$$\frac{\gamma}{g} = \lambda\underline{\omega} + \mu, \tag{31}$$

and

$$\lambda(1-2\tau)(A_\theta - \underline{\omega}) = \tau A_\theta + (1-\tau)\underline{\omega} - \mu. \tag{32}$$

There are three cases to consider.

Case 1. Assume first that the budget constraint is not binding, so that $\lambda = 0$. In this case (31) implies that $\mu = \frac{\gamma}{g} > 0$ and hence there is no unemployment. Substituting this expression for μ into (32), we obtain

$$\tau A_\theta + (1-\tau)\underline{\omega} = \frac{\gamma}{g}. \tag{33}$$

Combining this equation with the resource constraint and solving, we find that the solution is $(\tau_\theta^o, g_\theta^o)$ as described in (15) and (16). Recalling that $r_\theta^o = R_\theta(\tau_\theta^o,\underline{\omega}) - \underline{\omega}g_\theta^o$, we need that $r \leq r_\theta^o$ for the budget constraint not to be binding.

Case 2. Assume next that both the budget constraint and the resource constraints are binding, so that $\lambda > 0$ and $\mu > 0$. In this case, we have that

$$R_\theta(\tau,\underline{\omega}) - \underline{\omega}g = r, \tag{34}$$

and

$$g + \frac{x_\theta(\tau)}{A} = n_w. \tag{35}$$

Substituting (35) into (34), we obtain

$$R_\theta(\tau,\underline{\omega}) - \underline{\omega}\left(n_w - \frac{x_\theta(\tau)}{A}\right) = r. \tag{36}$$

Assuming that (36) has a solution, it will have two solutions $\tau_\theta^-(r)$ and $\tau_\theta^+(r)$, which correspond to the points illustrated in Fig. 1.C. The associated public good levels, $g_\theta^-(r)$ and $g_\theta^+(r)$, are then obtained from (35).

It remains to describe when (36) has a solution and also whether $\left(\tau_\theta^-(r), g_\theta^-(r)\right)$ or $\left(\tau_\theta^+(r), g_\theta^+(r)\right)$ provide a higher value of the objective function. In order for (36) to have a solution, the budget line must lie above the resource constraint for some range of taxes. Let τ_θ^* denote the tax rate at which the slope of the budget line is equal to the slope of the full employment line and let $g_\theta^*(r)$ denote the level of public good that satisfies the budget constraint (34) given this tax rate. Now define r_θ^* to be that revenue requirement at which the point $(\tau_\theta^*, g_\theta^*(r))$ is tangent to the full employment line. Then, (36) has a solution if and only if $r \leq r_\theta^*$. Moreover, $\tau_\theta^-(r_\theta^*) = \tau_\theta^+(r_\theta^*) = \tau_\theta^*$.

Turning to the issue of whether $\left(\tau_\theta^-(r), g_\theta^-(r)\right)$ or $\left(\tau_\theta^+(r), g_\theta^+(r)\right)$ provide a higher value of the objective function, assume that $r \in (r_\theta^o, r_\theta^*)$. There are two possibilities: i) $\tau_\theta^o < \tau_\theta^*$, and ii) $\tau_\theta^o > \tau_\theta^*$. In case i), the optimal policy combination ignoring the budget constraint $(\tau_\theta^o, g_\theta^o)$ is to the left of $\left(\tau_\theta^-(r), g_\theta^-(r)\right)$. Since the objective function is concave, the optimal choice is $\left(\tau_\theta^-(r), g_\theta^-(r)\right)$. In this range, taxes and public goods are increasing in r. In case ii), the optimal policy combination ignoring the budget constraint is on the right of $\left(\tau_\theta^+(r), g_\theta^+(r)\right)$. Again, concavity of the objective function implies that the optimal choice is $\left(\tau_\theta^+(r), g_\theta^+(r)\right)$. In this range, taxes and public goods are decreasing in r.

It remains to determine when τ_θ^o is larger or smaller than τ_θ^*. This is equivalent to knowing whether g_θ^o is larger or smaller than $g_\theta^*(r_\theta^*)$. We know g_θ^o from (15), so we just need to compute $g_\theta^*(r_\theta^*)$. By definition, the tax rate τ_θ^* is such that the slope of the budget line is the same as the slope of the resource constraint. This means that

$$\frac{n_e(A_\theta - \underline{\omega})}{\xi} = \frac{n_e(1 - 2\tau_\theta^*)(A_\theta - \underline{\omega})^2}{\xi\underline{\omega}}, \tag{37}$$

which implies that $\tau_\theta^* = \frac{A_\theta - 2\underline{\omega}}{2(A_\theta - \underline{\omega})}$. We know that the revenue requirement r_θ^* is such that the associated budget line is just tangent to the resource constraint at tax rate τ_θ^*. Thus,

$$g_\theta^*(r_\theta^*) + \frac{x_\theta(\tau_\theta^*)}{A} = n_w. \tag{38}$$

Using (8), this implies that

$$g_\theta^*(r_\theta^*) = n_w - \frac{n_e A_\theta}{2\xi}. \tag{39}$$

Thus,

$$g_\theta^o - g_\theta^*(r_\theta^*) = \frac{\sqrt{(A_\theta n_e - \xi n_w)^2 + 4\xi n_e \gamma} - \xi n_w}{2\xi} \tag{40}$$

and we have g_θ^o is larger or smaller than $g_\theta^*(r_\theta^*)$ as γ is larger or smaller than $\frac{A_\theta}{2}\left(n_w - \frac{n_e A_\theta}{2\xi}\right)$.

We conclude that a necessary condition to be in Case 2 is that $r \in (r_\theta^o, r_\theta^*]$ and that in Case 2 the solution is $\left(\tau_\theta^-(r), g_\theta^-(r)\right)$ if (18) is satisfied and $\left(\tau_\theta^+(r), g_\theta^+(r)\right)$ otherwise. We will return to provide a necessary and sufficient condition to be in Case 2 after analyzing Case 3.

Case 3. Finally, assume that only the budget constraint binds, so that $\lambda > 0$ and $\mu = 0$. Substituting (31) into (32), we obtain

$$g = \frac{\gamma(1-2\tau)(A_\theta - \underline{\omega})}{\underline{\omega}\left(\tau(A_\theta - \underline{\omega}) + \underline{\omega}\right)}. \tag{41}$$

Substituting (41) into the budget constraint (34), we obtain:

$$\tau n_e(1-\tau)(A_\theta - \underline{\omega})^2/\xi - \left(\frac{\gamma(1-2\tau)(A_\theta - \underline{\omega})}{\tau(A_\theta - \underline{\omega}) + \underline{\omega}}\right) = r \tag{42}$$

This equation has a unique solution $\widehat{\tau}_\theta(r)$ in the relevant range for τ, i.e. $[0, 1/2]$. Since the right hand side of (42) is always increasing for τ less than $1/2$, $\widehat{\tau}_\theta(r)$ is increasing in r. The associated value of g, $\widehat{g}_\theta(r)$, is obtained from (41). Since the right hand side of (41), is decreasing in τ, $\widehat{g}_\theta(r)$ is decreasing in r. Furthermore, note that unemployment is an increasing function of τ and a decreasing function of g, so it is increasing in r as well.

Now define the revenue requirement r_θ^{**} to be such that the resource constraint is satisfied with equality at the policies $(\widehat{g}_\theta(r), \widehat{\tau}_\theta(r))$; that is,

$$\widehat{g}_\theta(r_\theta^{**}) + \frac{x_\theta(\widehat{\tau}_\theta(r_\theta^{**}))}{A} = n_w.$$

It is clear that this revenue requirement exists, is unique, and satisfies $r_\theta^{**} \in (r_\theta^o, r_\theta^*]$. Moreover, if $r < r_\theta^{**}$, then it must be the case that the resource constraint binds while if $r > r_\theta^{**}$, then the resource constraint does not bind. Thus, as claimed in the Proposition, we are in Case 2 if $r \in (r_\theta^o, r_\theta^{**}]$ and Case 3 if $r > r_\theta^{**}$. ■

6.2 Proof of Proposition 2

Using the Envelope Theorem and the first order condition for b', it is straightforward to show that for each cost state θ and any initial debt level b, the optimal level of borrowing $b'_\theta(b)$ is such that:

$$1 + \lambda_\theta(b) - \psi_\theta(b) = 1 + E\left[\lambda_{\theta'}(b'_\theta(b))\right], \tag{43}$$

where $\lambda_\theta(b)$ and $\psi_\theta(b)$ are, respectively, the Lagrange multipliers of the budget constraint and of the upperbound on debt in (20). We now proceed in two steps.

Step 1. We first prove that for each cost state θ and any initial debt level b such that $b < \overline{b}$ there is an $\phi(\theta, b) > 0$ such that $b'_\theta(b) < \overline{b} - \phi(\theta, b)$. Consider the optimal policy $\{\tau_\theta(b), g_\theta(b), b'_\theta(b)\}$ with associated Lagrange multipliers $\lambda_\theta(b)$, $\psi_\theta(b)$, and $\mu_\theta(b)$, where $\mu_\theta(b)$ is the multiplier on the resource constraint in (20). As noted in the text, if $(\tau^s_\theta(r), g^s_\theta(r))$ denote the optimal static policies described in Proposition 1, it is clear that $(\tau_\theta(b), g_\theta(b))$ will equal $(\tau^s_\theta(r_\theta(b)), g^s_\theta(r_\theta(b)))$ where $r_\theta(b) = (1+\rho)b - b'_\theta(b)$ is the revenue requirement implied by the optimal borrowing level. Assume, by contradiction, that $b'_\theta(b)$ is arbitrarily close to $\overline{b}$; that is, $b'_\theta(b) = \overline{b} - \zeta$, where ζ is arbitrarily small. There are three cases to consider.

Case 1.1. Assume that $\lambda_\theta(b) > 0$ and $\mu_\theta(b) > 0$. Suppose first that (18) is satisfied. In this case, from Proposition 1, taxes and public goods are given by $(\tau^-_\theta(r(b,\zeta)), g^-_\theta(r(b,\zeta)))$, where $r(b,\zeta) = (1+\rho)b - \overline{b} + \zeta$. Combining the first order conditions (31) and (32), we have:

$$\lambda_\theta(b,\zeta) = \frac{\tau^-_\theta(r(b,\zeta)A_\theta + (1-\tau^-_\theta(r(b,\zeta))\underline{\omega} - \frac{\gamma}{g^-_\theta(r(b,\zeta))}}{(1-2\tau^-_\theta(r(b,\zeta))(A_\theta - \underline{\omega}) - \underline{\omega}} \tag{44}$$

Consider now $\lambda_{\theta'}(\overline{b}-\zeta)$, for $\theta' \in \{H, L\}$. As $\zeta \to 0$, we must have $\tau_H(\overline{b}-\zeta) \to 1/2$ and $g_H(\overline{b}-\zeta) \to 0$. By Assumption 1, this implies that $\mu_H(\overline{b}-\zeta) = 0$ for ζ sufficiently small.[33] We therefore have from (32) that

$$\lambda_H(\overline{b}-\zeta) = \frac{\tau_H(\overline{b}-\zeta)A_H + (1-\tau_H(\overline{b}-\zeta))\underline{\omega}}{(1-2\tau_H(\overline{b}-\zeta))(A_H - \underline{\omega})}$$

for ζ small. It follows that as $\zeta \to 0$, $\lambda_H(\overline{b}-\zeta)$ becomes arbitrarily large. Since $\lambda_L(\overline{b}-\zeta) \geq 0$, this implies that $E\left[\lambda_{\theta'}(\overline{b}-\zeta)\right]$ becomes arbitrarily large. Since $\lim_{\zeta\to 0} r(b,\zeta) < \rho\overline{b}$, on the contrary, condition (44) shows that $\lambda_\theta(b,\zeta)$ is bounded as $\zeta \to 0$.[34] This generates a contradiction since by (43) we must have $\lambda_\theta(b) \geq E\left[\lambda_{\theta'}(b'_\theta(b))\right]$. The case in which (18) is not satisfied is completely analogous.

Case 1.2. Assume that $\lambda_\theta(b) > 0$ and $\mu_\theta(b) = 0$. In this case, from Proposition 1, taxes and public goods are given by $(\widehat{\tau}_\theta(r_\theta(b)), \widehat{g}_\theta(r_\theta(b)))$. We can write $g_\theta(b) = f(\tau_\theta(b))$ where $f(\tau)$ is a continuous function defined by the right hand side of equation (41). Since $\lambda_\theta(b) > 0$, we must also

[33] This follows from the fact that Assumption 1 implies $n_w > n_e(A_H - \underline{\omega})/2\xi$.

[34] If $\lambda_\theta(b,\zeta) \to \infty$ as $\zeta \to 0$, then τ would have to converge to 1/2 and g to 0: but then neither the resource constraint, nor the budget constraint would be binding, a contradiction.

have:

$$B_\theta(\tau_\theta(b), f(\tau_\theta(b)), b'_\theta(b), b, \underline{\omega}) = 0. \tag{45}$$

It follows that if $b'_\theta(b) = \overline{b} - \zeta$, we can express all the policy choices as a function of ζ. Thus, setting $b'_\theta(b) = \overline{b} - \zeta$, we have $\tau_\theta(b) = \tau(\zeta)$ where $\tau(\zeta)$ solves (45) and $g_\theta(b) = f(\tau(\zeta)) = g(\zeta)$. Note that as $\zeta \to 0$, we have $\tau(\zeta) \to \widetilde{\tau} < 1/2$. For if $\tau(\zeta) \to 1/2$, then $g(\zeta) \to 0$ and (45) would not be satisfied since $b < \overline{b}$. Moreover, $\tau(\zeta) \to \widetilde{\tau}$ implies $g(\zeta) \to \widetilde{g} > 0$. From (32) and (43), we have that:

$$\lambda_\theta(b) = \frac{\tau_\theta(b)A_\theta + (1-\tau_\theta(b))\underline{\omega}}{(1-2\tau_\theta(b))(A_\theta - \underline{\omega})} \geq E\left[\lambda_{\theta'}(\overline{b}-\zeta)\right] \geq \alpha\left[\frac{\tau_H(\overline{b}-\zeta)A_H + (1-\tau_H(\overline{b}-\zeta))\underline{\omega}}{(1-2\tau_H(\overline{b}-\zeta))(A_\theta - \underline{\omega})}\right], \tag{46}$$

where the last equality follows from the fact that, as in Case 1.1, we must have $\tau_H(\overline{b}-\zeta) \to 1/2$ and $g_H(\overline{b}-\zeta) \to 0$ as $\zeta \to 0$: and this implies that $\mu_H(\overline{b}-\zeta) = 0$ for ζ sufficiently small. As in Case 1.1, the right hand side of (46) diverges to infinity, while the left hand side converges to a finite value: so we again have a contradiction.

Case 1.3. Assume $\lambda_\theta(b) = 0$. In this case (43) implies that $\psi_\theta(b) = 0$ and $\mu_\theta(b) > 0$ as well. This implies that taxes and public goods are given by $(\tau^o_\theta, g^o_\theta)$. The first order condition with respect to b' is $1 = E\left[\lambda_{\theta'}(\overline{b}-\zeta)\right]$, which is impossible since by an argument similar to the argument of the previous two cases, the right hand side becomes arbitrarily large as $\zeta \to 0$.

Step 2. Step 1 implies that $b'_\theta(b) < \overline{b}$ and that $\psi_\theta(b) = 0$ for any $b < \overline{b}$. From (43) we therefore conclude that the Lagrange multiplier $\lambda_\theta(b)$ is a non-negative martingale. Defining the sequences $\langle \tau_t, g_t, b'_t \rangle$ as in the text, Corollary 2 of Shiryaev (1991) (p. 508) implies that $\Pr\left(\lim_{t\to\infty} \lambda_{\theta_t}(b'_{t-1}) \text{ exists}\right) = 1$. Define $\underline{b} = r^o_H/\rho$ to be the level of assets that in steady state generate interest earnings just sufficient to cover the discrepancy between $\underline{\omega} g^o_H$ and $R_H(\tau^o_H, \underline{\omega})$. When $b \leq \underline{b}$, by holding asset levels constant the government can achieve full employment and the optimal output mix in both the high and low cost state in the current and all future periods. So $\lambda_\theta(b) = 0$ for any θ if and only if $b \leq \underline{b}$. Assume by contradiction that $\Pr(\lim_{t\to\infty} r_t > r^o_H) > 0$. In this case, there must be an $\varepsilon > 0$, and a set of sequences $\langle r_t \rangle$ with positive probability such that, for each sequence in the set, we can find a subsequence $\langle r_{n(t)} \rangle$ with $r_{n(t)} > r^o_H + \varepsilon$ for any arbitrarily large $n(t)$. Note that $r_t > r^o_H + \varepsilon$ implies that $\lambda_H(b'_{t-1}) > 0$ and $\lambda_H(b'_{t-1}) > \lambda_L(b'_{t-1})$. Along these sequences, therefore, $\lambda_H(b'_{n(t)-1}) - \lambda_L(b'_{n(t)-1}) > \eta$ for some positive η: this contradicts the fact that $\lambda_{\theta_{n(t)}}(b'_{n(t)-1})$ converges with probability one. We conclude that $\Pr(\lim_{t\to\infty} r_t \leq r^o_H) = 1$. The result now follows from Proposition 1. ■

6.3 Proof of Proposition 3

We will demonstrate the existence of a pair of concave value functions $V_H(b)$ and $V_L(b)$ that satisfy (22) given the policy functions they generate via (21). Note first that in political equilibrium, the minimum winning coalition will always choose a higher debt level than $\underline{b} \equiv r_H^o/\rho$. If the debt level were below $\underline{b}$, the mwc could marginally increase debt, use the proceeds to finance transfers, and increase its current period payoff by q (see (21)). The cost of such an action would be going into the next period with a marginally higher debt level. But the only effect of this would be to reduce the transfers received by next period's mwc (see Propositions 4 and 5 below). Since members of the current mwc will not necessarily belong to next period's mwc, the discounted expected reduction in future transfer payoffs is $1 < q$. Thus, increasing debt is optimal.

Given this observation, we look for a pair of equilibrium value functions in the compact space of continuous and concave functions defined on the interval $\left[\underline{b}, \overline{b}\right]$. Let F be the metric space of real valued continuous and bounded, weakly concave functions of b in $\left[\underline{b}, \overline{b}\right]$ endowed with the sup norm, $\|f\| = \sup_{b \in \left[\underline{b}, \overline{b}\right]} |f|$. Let F^2 be the Cartesian product of two such spaces endowed with the maximum norm, $\left\|f^2\right\| = \max\left\{\left\|f_1^2\right\|, \left\|f_2^2\right\|\right\}$, for any $f^2 \in F^2$.

As noted in the paragraph following the statement of Proposition 3, the policies in a political equilibrium solve the following problem:

$$\max_{(\tau,g,b')} \left\{ \begin{array}{c} b' - (1+\rho)b + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{A n_e}\right)^2}{2} + \gamma \ln g + (q-1)\, B_\theta(\tau, g, b', b, \underline{\omega}) + \beta E V_{\theta'}(b') \\ s.t.\ B_\theta(\tau, g, b', b, \underline{\omega}) \geq 0,\ g + \frac{x_\theta(\tau)}{A} \leq n_w\ \&\ b \in \left[\underline{b}, \overline{b}\right] \end{array} \right\}. \tag{47}$$

To compact notation let $p = (\tau, g, b')$ denote a generic policy and let $P = [\underline{\tau}, 1] \times [0, \overline{g}] \times \left[\underline{b}, \overline{b}\right]$ denote the feasible policy space, where $\overline{g}$ is some sufficiently large upper bound on public goods and $\underline{\tau}$ is some sufficiently low lower bound on taxes. Note that we can impose an upper bound on public good levels without loss of generality since in every period revenues are bounded by $\max_\tau R_L(\tau, \underline{\omega}) + \overline{b}$; similarly there is no loss of generality in assuming that the tax rate is bounded below since it would never be optimal nor feasible to provide unbounded subsidies to the entrepreneurs. Define a *policy function* to be a function $p_\theta(b)$ which associates a policy with any given initial debt level b and cost state θ. Let $P_\theta(b; V) \subseteq P$ be the set of optimal policies for (47) when the initial debt level is b, the state is θ, and the value functions are $V = (V_H, V_L)$. Let $P_\theta(V)$ be the set of optimal policy functions for the problem with value functions V; that is, $p_\theta(b) \in P_\theta(V)$

if and only if $p_\theta(b) \in P_\theta(b; V)$ for all $b \in [\underline{b}, \overline{b}]$.

Now, for any $V \in F^2$, define $P_\theta^*(V) \subset P_\theta(V)$ to be the subset of optimal policy functions which are (i) continuous in b, and, (ii) which generate a net of transfer budget surplus function $B_\theta(\tau_\theta(b), g_\theta(b), b'_\theta(b), b, \underline{\omega})$ that is weakly convex in b. Observe that, when viewed as a function of V, $P_\theta^*(V)$ is a correspondence from F^2 into the set of policy functions.

Lemma A.1. *For each state θ, the correspondence $P_\theta^*(V)$ is non empty, compact, and convex valued.*

Proof. Available from the authors on request. ■

For a given $V \in F^2$ and cost state θ, let $\Upsilon_\theta(V)$ be defined by:

$$\Upsilon_\theta(V) = \left\{ \widetilde{V}(b) \;\middle|\; \begin{array}{c} \exists\, p(b) \in P_\theta^*(V) \; s.t. \; \widetilde{V}(b) = \left[\begin{array}{c} b'(b) - (1+\rho)b + x_\theta(\tau(b)) \left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau(b))}{A n_e}\right)^2}{2} \\ +\gamma \ln g(b) + \beta E V_{\theta'}(b'(b)) \end{array} \right] \\ for \; (\tau(b), g(b), b'(b)) = p(b) \end{array} \right\}.$$

This expression defines a correspondence from F^2 into the set of real-valued functions defined on $[\underline{b}, \overline{b}]$:

Lemma A.2. *For each cost state θ, $\Upsilon_\theta(V)$ is a non empty, convex, and compact valued correspondence from F^2 into F with a closed graph.*

Proof. Available from the authors on request. ■

Define the correspondence from F^2 into F^2:

$$T(V) = \left\{ (\widetilde{V}_H(b), \widetilde{V}_L(b)) \,\middle|\, \widetilde{V}_\theta(b) \in \Upsilon_\theta(V) \quad \theta \in \{H, L\} \right\}.$$

We have:

Lemma A.3. *The correspondence $T(V)$ has a fixed point $V^* = T(V^*)$.*

Proof. It can be verified that Lemma A.2 implies that $T(V)$ is a non empty, compact, and convex-valued correspondence form F^2 to F^2 with a closed graph. The result therefore follows from the Glicksberg-Fan Theorem (see Theorem 9.2.2 in Smart (1974)). ■

Let $(V_H(b), V_L(b))$ be a fixed point of the correspondence $T(V)$. Then $V_H(b)$ and $V_L(b)$ are concave functions that by construction satisfy (22) given the policy functions they generate via (21). Proposition 3 is therefore established. ■

6.4 Proof of Propositions 4 and 5

Given the discussion in the text, it suffices to show that the solution to Problem (26) when the budget constraint is not binding, which is denoted $(\tau_\theta^q, g_\theta^q)$, is given by (27) when $q < q_\theta^*$ and by (30) when $q > q_\theta^*$. The Lagrangian for Problem (26) (ignoring the budget constraint) is

$$L = x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + (q-1)\left(R_\theta(\tau,\underline{\omega}) - \underline{\omega} g - r\right) - qr + \mu\left(n_w - g - \frac{x_\theta(\tau)}{A}\right),$$

where μ is the multiplier on the resource constraint. Using the expressions for $x_\theta(\tau)$ and $R_\theta(\tau,\underline{\omega})$ in (8) and (9), the first order conditions with respect to g and τ respectively are

$$\frac{\gamma}{g} = (q-1)\underline{\omega} + \mu, \tag{48}$$

and

$$(q-1)(1-2\tau)(A_\theta - \underline{\omega}) = \tau A_\theta + (1-\tau)\underline{\omega} - \mu. \tag{49}$$

It is easy to show that if $\mu = 0$, these equations imply that the solution is given by (30). It follows that the resource constraint is not binding if at these values of $(\tau_\theta^q, g_\theta^q)$, $g_\theta^q + \frac{x_\theta(\tau_\theta^q)}{A} \leq n_w$. This implies:

$$n_e\left[\frac{q(A_\theta - \underline{w}) + \underline{w}}{\xi(2q-1)}\right] + \frac{\gamma}{(q-1)\,\underline{w}} \leq n_w. \tag{50}$$

Note that the left hand side of (50) is decreasing in q, so it is satisfied if and only if $q > q_\theta^*$, where q_θ^* is defined by (29). If the resource constraint is binding, then (48) implies that $\mu = \frac{\gamma}{g} - (q-1)\underline{\omega} > 0$. Substituting this expression for μ into (49), we obtain

$$\tau A_\theta + (1-\tau)\underline{\omega} = \frac{\gamma}{g} + (q-1)\left[(1-2\tau)(A_\theta - \underline{\omega}) - \underline{\omega}\right]. \tag{51}$$

Combining this equation with the resource constraint and solving we find that the solution is as described in (27). ■

6.5 Proof of Proposition 6

The proof is broken into three parts. In Section 6.5.1 we characterize b^q - the lower bound of the equilibrium debt distribution. In Section 6.5.2 we prove that debt behaves in a counter-cyclical way. In Section 6.5.3 we prove that a non-degenerate stable distribution exists and has full support in $\left[b^q, \overline{b}\right)$.

6.5.1 The lowerbound b^q

Consider problem (25). When the budget constraint is not binding, the mwc will choose a debt level from the set[35]

$$\mathcal{X}(V) = \arg\max_{b' \in [\underline{b}, \overline{b}]} \left\{ qb' + \beta E V_{\theta'}(b') \right\}.$$

We will show that $\mathcal{X}(V)$ consists of just a single point. We first need:

Lemma A.4. $r_L^q > r_H^q$.

Proof. Available from the authors on request. ■

We can now show that:

Lemma A.5. *In any equilibrium, the set $\mathcal{X}(V)$ is a singleton.*

Proof. Available from the authors on request. ■

Given Lemma A.5, we define b^q to be the unique element of the set $\mathcal{X}(V)$. We also define b_θ^q to be the value of debt such that the triple $(\tau_\theta^q, g_\theta^q, b^q)$ satisfies the constraint that $B_\theta(\tau_\theta^q, g_\theta^q, b^q, b_\theta^q, \underline{w})$ equal 0. This is given by:

$$b_\theta^q = \frac{r_\theta^q + b^q}{1+\rho}. \tag{52}$$

Then, if the debt level b is such that $b \leq b_\theta^q$ the tax-public good-debt triple is $(\tau_\theta^q, g_\theta^q, b^q)$ and the mwc uses the budget surplus $B_\theta(\tau_\theta^q, g_\theta^q, b^q, b_\theta^q, \underline{\omega})$ to finance transfers. If $b > b_\theta^q$ the budget constraint binds so that no transfers are given. Tax revenues net of public good costs strictly exceed r_θ^q and the debt level strictly exceeds b^q. In this case, the policies solve the problem

$$\max_{(\tau,g,b')} \left\{ \begin{array}{c} b' - (1+\rho)b + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + \beta E V_{\theta'}(b') \\ s.t.\ B_\theta(\tau, g, b', b, \underline{\omega}) \geq 0,\ g + \frac{x_\theta(\tau)}{A} \leq n_w \ \&\ b \in [\underline{b}, \overline{b}] \end{array} \right\} \tag{53}$$

Note also that by Lemma A.4., it must be the case that $b_L^q > b_H^q$.

Further information on the debt level b^q can be obtained by using a first order condition to characterize it. However, before we can do this, we must first establish that the value function is differentiable. We have:

[35] As explained in the proof of Proposition 3, there is no loss of generality in assuming that $b' \geq \underline{b}$.

Lemma A.6. *(i) If $q > q_\theta^*$, the equilibrium value function $V_\theta(b)$ is differentiable for all $b \neq b_\theta^q$. Moreover,*

$$-\beta V_\theta'(b) = \begin{cases} 1 & if\ b < b_\theta^q \\ 1 + \frac{\tau_\theta(b)A_\theta + (1-\tau_\theta(b))\underline{\omega}}{(1-2\tau_\theta(b))(A_\theta - \underline{\omega})} & if\ b > b_\theta^q \end{cases}.$$

(ii) If $q < q_\theta^*$, there exists a unique debt level $b_\theta^{**} \in (b_\theta^q, \overline{b}]$ such that the resource constraint is binding if and only if $b \leq b_\theta^{**}$. The equilibrium value function $V_\theta(b)$ is differentiable for all $b \neq b_\theta^q$ and

$$-\beta V_\theta'(b) = \left\{ \begin{array}{cc} 1 & if\ b < b_\theta^q \\ 1 + \frac{\tau_\theta(b)A_\theta + (1-\tau_\theta(b))\underline{\omega} - \frac{\gamma}{g_\theta(b)}}{(1-2\tau_\theta(b))(A_\theta - \underline{\omega}) - \underline{\omega}} & b \in (b_\theta^q, b_\theta^{**}) \\ 1 + \frac{\tau_\theta(b)A_\theta + (1-\tau_\theta(b))\underline{\omega}}{(1-2\tau_\theta(b))(A_\theta - \underline{\omega})} & b \geq b_\theta^{**} \end{array} \right\}.$$

Proof. (i) Suppose first that $q > q_\theta^*$. From the discussion presented above, we know that if the debt level b is such that $b \leq b_\theta^q$ the optimal policies are $(\tau_\theta^q, g_\theta^q, b^q)$, and, if $b > b_\theta^q$, the budget constraint $B_\theta(\tau, g, b', b, \underline{\omega}) \geq 0$ will be binding and the policies solve (53). Moreover, we know from Proposition 5 that the resource constraint will not be binding. Consider some debt level b_o. Assume first that $b_o < b_\theta^q$. Then, we know from (22) that in a neighborhood of b_o it must be the case that

$$V_\theta(b) = b^q - (1+\rho)b + x_\theta(\tau_\theta^q)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau_\theta^q)}{An_e}\right)^2}{2} + \gamma \ln g_\theta^q + \beta E V_\theta(b^q).$$

Thus, it is immediate that the value function $V_\theta(b)$ is differentiable at b_o and that

$$V_\theta'(b_o) = -(1+\rho) = -1/\beta.$$

Assume now that $b_o > b_\theta^q$. Consider the function

$$\varphi_\theta(b) = \max_{(\tau,g)} \left\{ \begin{array}{c} b_\theta'(b_o) - (1+\rho)b + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + \beta E V_{\theta'}(b_\theta'(b_o)) \\ s.t.\ B_\theta(\tau, g, b_\theta'(b_o), b, \underline{\omega}) = 0. \end{array} \right\} \quad (54)$$

Since the equilibrium policies are such that the budget constraint binds and the resource constraint does not bind, it follows that $V_\theta(b_o) = \varphi_\theta^o(b_o)$ and $V_\theta(b) \geq \varphi_\theta^o(b)$ for all b in a neighborhood of b_o. By the Envelope Theorem, the function $\varphi_\theta(b)$ is differentiable in b and its derivative is equal to $-(1+\rho)[1+\lambda_\theta^o(b)]$, where $\lambda_\theta^o(b)$ is the Lagrange multiplier on the constraint

$B_\theta(\tau, g, b'_\theta(b_o), b, \underline{\omega}) = 0$ in (54) at b. It is also the case that $\varphi^o_\theta(b)$ is concave. To see this note that we may write:

$$\varphi_\theta(b) = \max_{(\tau,g)} \left\{ \begin{array}{c} b'_\theta(b_o) - (1+\rho)b + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + \beta E V_{\theta'}(b'_\theta(b_o)) \\ s.t.\ B_\theta(\tau, g, b'_\theta(b_o), b, \underline{\omega}) \geq 0. \end{array} \right\} \quad (55)$$

The objective function in (55) is concave in τ, g and b, and the constraint set is convex in τ, g and b: so by a standard argument $\varphi_\theta(b)$ is concave. It now follows from Theorem 4.10 of Stokey, Lucas and Prescott (1989) that $V_\theta(b)$ is differentiable at b_o with derivative $V'_\theta(b_o) = \varphi'_\theta(b_o) = -(1+\rho)[1+\lambda^o_\theta(b_o)] = -[1+\lambda^o_\theta(b_o)]/\beta$. To complete the proof, consider the first order conditions for (54) at b_o:

$$\frac{\gamma}{g^o_\theta(b_o)} = \lambda^o_\theta(b_o)\underline{\omega} \quad (56)$$

and

$$\lambda^o_\theta(b_o)(1-2\tau^o_\theta(b_o))(A_\theta - \underline{\omega}) = \tau^o_\theta(b_o)A_\theta + (1-\tau^o_\theta(b_o))\underline{\omega}. \quad (57)$$

The second condition implies

$$\lambda^o_\theta(b_o) = \frac{\tau_\theta(b_o)A_\theta + (1-\tau_\theta(b_o))\underline{\omega}}{(1-2\tau_\theta(b_o))(A_\theta - \underline{\omega})}$$

where we are using the fact that the solution of (54) at b_o, $(\tau^o_\theta(b_o), g^o_\theta(b_o))$, must equal the equilibrium policies $(\tau_\theta(b_o), g_\theta(b_o))$. We conclude that

$$-\beta V'_\theta(b_o) = 1 + \frac{\tau_\theta(b_o)A_\theta + (1-\tau_\theta(b_o))\underline{\omega}}{(1-2\tau_\theta(b_o))(A_\theta - \underline{\omega})}. \quad (58)$$

(ii) Suppose now that $q < q^*_\theta$. Then we know from Proposition 4 that the resource constraint is binding for $b \leq b^q_\theta$. Consider the problem

$$\max_{(\tau,g,b')} \left\{ \begin{array}{c} b' - (1+\rho)b + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + (q-1)B_\theta(\tau, g, b', b, \underline{\omega}) + \beta E V_\theta(b') \\ s.t.\ B_\theta(\tau, g, b', b, \underline{\omega}) \geq 0 \ \&\ b \in [\underline{b}, \overline{b}] \end{array} \right\}. \quad (59)$$

This is the mwc's problem but ignoring the resource constraint. Let $(\widetilde{\tau}_\theta(b), \widetilde{g}_\theta(b))$ denote the optimal tax and public good levels for this problem. It is easy to show that $\widetilde{\tau}_\theta(b)$ is non-decreasing and $\widetilde{g}_\theta(b)$ is non-increasing in b, implying that $\widetilde{g}_\theta(b) + \frac{x_\theta(\widetilde{\tau}_\theta(b))}{A}$ is non-increasing in b. From Proposition 4, we know that $\widetilde{g}_\theta(b^q_\theta) + \frac{x_\theta(\widetilde{\tau}_\theta(b^q_\theta))}{A} > n_w$. If $\widetilde{g}_\theta(\overline{b}) + \frac{x_\theta(\widetilde{\tau}_\theta(\overline{b}))}{A} < n_w$, define b^{**}_θ to be the minimal level of b such that $\widetilde{g}_\theta(b)+\frac{x_\theta(\widetilde{\tau}_\theta(b))}{A}\leq n_w$. Otherwise let $b_\theta^{**}=\overline{b}$. Then the resource constraint in the mwc's problem is binding if and only if $b\leq b_\theta^{**}$.

Turning to differentiability, consider some debt level b_o. If $b_o<b_\theta^q$ or $b_o>b_\theta^{**}$, the argument follows exactly that in part (i). Suppose that $b_o\in(b_\theta^q,b_\theta^{**})$. It follows that both the budget and resource constraints are binding in a neighborhood of b_o. Consider the function

$$\varphi_\theta\left(b\right)=\max_{(\tau,g)}\left\{\begin{array}{c} b_\theta'(b_o)-(1+\rho)b+x_\theta(\tau)\left(\frac{A_\theta}{A}\right)-n_e\xi\frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2}+\gamma\ln g+\beta EV_{\theta'}(b_\theta'(b_o)) \\ s.t.\ B_\theta(\tau,g,b_\theta'(b_o),b,\underline{\omega})=0,\ g+\frac{x_\theta(\tau)}{A}=n_w. \end{array}\right\} \quad (60)$$

Clearly, $V_\theta(b_o)=\varphi_\theta^o\left(b_o\right)$ and $V_\theta(b)\geq\varphi_\theta^o\left(b\right)$ for all b in a neighborhood of b_o. The function $\varphi_\theta\left(b\right)$ is differentiable in b and its derivative is equal to $-\left(1+\rho\right)\left[1+\lambda_\theta^o\left(b\right)\right]$, where $\lambda_\theta^o\left(b\right)$ is the Lagrangian multiplier of the constraint $B_\theta(\tau,g,b_\theta'(b_o),b,\underline{\omega})=0$ in (60) at b. By a similar argument to that just used, $\varphi_\theta\left(b\right)$ is concave. It again follows from Theorem 4.10 of Stokey, Lucas and Prescott (1989) that $V_\theta(b)$ is differentiable at b_o with derivative $V_\theta'(b_o)=\varphi_\theta'\left(b_o\right)=-\left(1+\rho\right)\left[1+\lambda_\theta^o\left(b_o\right)\right]=-\left[1+\lambda_\theta^o\left(b_o\right)\right]/\beta$. To complete the proof, consider the first order conditions of (60) at b_o:

$$\frac{\gamma}{g_\theta^o\left(b_o\right)}=\lambda_\theta^o\left(b_o\right)\underline{\omega}+\mu_\theta^o\left(b_o\right) \quad (61)$$

and

$$\lambda_\theta^o\left(b_o\right)(1-2\tau_\theta^o\left(b_o\right))(A_\theta-\underline{\omega})=\tau_\theta^o\left(b_o\right)A_\theta+(1-\tau_\theta^o\left(b_o\right))\underline{\omega}-\mu_\theta^o\left(b_o\right), \quad (62)$$

where $\mu_\theta^o\left(b_o\right)$ is the Lagrange multiplier of the resource constraint. Solving the system, we have:

$$\lambda_\theta\left(b_o\right)=\frac{\tau_\theta\left(b_o\right)A_\theta+\left(1-\tau_\theta\left(b_o\right)\right)\underline{\omega}-\frac{\gamma}{g_\theta\left(b_o\right)}}{\left(1-2\tau_\theta\left(b_o\right)\right)\left(A_\theta-\underline{\omega}\right)-\underline{\omega}}$$

where we are using the fact that the solution of (60) at b_o, $\left(\tau_\theta^o\left(b_o\right),g_\theta^o\left(b_o\right)\right)$ equals the equilibrium policies $\left(\tau_\theta\left(b_o\right),g_\theta\left(b_o\right)\right)$. We therefore conclude that:

$$-\beta V_\theta'(b_o)=1+\frac{\tau_\theta\left(b_o\right)A_\theta+\left(1-\tau_\theta\left(b_o\right)\right)\underline{\omega}-\frac{\gamma}{g_\theta\left(b_o\right)}}{\left(1-2\tau_\theta\left(b_o\right)\right)\left(A_\theta-\underline{\omega}\right)-\underline{\omega}}. \quad (63)$$

Finally, suppose that $b=b_\theta^{**}$. It is easy to see that in a neighborhood of b_θ^{**} the function $\varphi_\theta\left(b\right)$ defined in (60) satisfies the properties: $V_\theta(b_o)=\varphi_\theta^o\left(b_o\right)$ and $V_\theta(b)\geq\varphi_\theta^o\left(b\right)$. As noted above, moreover, the function $\varphi_\theta\left(b\right)$ is differentiable and concave in b; its derivative is equal to $-\left(1+\rho\right)\left[1+\lambda_\theta^o\left(b\right)\right]$, where $\lambda_\theta^o\left(b\right)$ is the Lagrangian multiplier of the constraint $B_\theta(\tau,g,b_\theta'(b_o),b,\underline{\omega})=0$ in (60) at b. It again follows from Theorem 4.10 of Stokey, Lucas and Prescott (1989) that

$V_\theta(b)$ is differentiable at b_θ^{**} with derivative $V_\theta'(b_\theta^{**}) = \varphi_\theta'(b_\theta^{**}) = -(1+\rho)[1+\lambda_\theta^o(b_\theta^{**})] = -[1+\lambda_\theta^o(b_\theta^{**})]/\beta$. The derivative is the same as above, with the difference that at b_θ^{**} we have $\mu_\theta^o(b_o) = 0$. So (58) is equal to (63) at b_θ^{**}. ■

We can now show:

Lemma A.7. $b^q \in [b_H^q, b_L^q]$.

Proof. From the definition of b^q, we know that if V_H and V_L are differentiable at b^q it must be the case that

$$q = -\beta E V_\theta'(b^q) \tag{64}$$

Assume first that $b^q < b_H^q$. Then, by Lemma A.6, (64) would imply $q = 1$, a contradiction. Assume next that $b^q > b_L^q$. This would imply that for each cost state θ, we have that $\tau_\theta(b^q) > \tau_\theta^q$. Using the first order conditions for τ_θ^q and the expressions in Lemma A.6, we can show that this implies $\beta V_\theta'(b^q) < -q$. This implies: $-\beta E V_\theta'(b^q) > q$: again a contradiction. We conclude that $b^q \in [b_H^q, b_L^q]$ as claimed. ■

We can use this result to establish the assertion in the proposition that $b^q > r_H^o/\rho$. Since $r_H^q > r_H^o$, we have from Lemma A.7 that

$$b^q \geq b_H^q = \frac{r_H^q + b^q}{1+\rho} > \frac{r_H^o + b^q}{1+\rho}$$

Multiplying this inequality through by $1+\rho$ yields the result.

6.5.2 Proof of countercyclical behavior

We begin with the following useful result.

Lemma A.8. *For all $b \in [b_H^q, \overline{b}]$ it is the case that $\lambda_H(b) > \lambda_L(b)$, where $\lambda_\theta(b)$ is the Lagrange multiplier on the budget constraint for the problem*

$$\max \left\{ \begin{array}{c} b' - b(1+\rho) + x_\theta(\tau)\left(\frac{A_\theta}{A}\right) - n_e \xi \frac{\left(\frac{x_\theta(\tau)}{An_e}\right)^2}{2} + \gamma \ln g + \beta E V_{\theta'}(b') \\ s.t.\ B_\theta(\tau, g, b', b, \underline{\omega}) \geq 0,\ g + \frac{x_\theta(\tau)}{A} \leq n_w \ \&\ b' \in [\underline{b}, \overline{b}] \end{array} \right\}.$$

Proof. Let $b \in [b_H^q, \overline{b}]$. Suppose, contrary to the claim, that $\lambda_L(b) \geq \lambda_H(b)$. Then, by the concavity of the value function, we know that $b_L'(b) \geq b_H'(b)$. Following the same basic steps as in the proof of Lemma A.4, we can show that:

$$R_L(\tau_L(b), \underline{\omega}) - \underline{\omega} g_L(b) > R_H(\tau_H(b), \underline{\omega}) - \underline{\omega} g_H(b). \tag{65}$$

Since $b \geq b_H^q$, moreover, we know that $\lambda_H(b) > 0$ and hence that the budget constraint is binding in the high cost state. Thus, if (65) holds, we have

$$R_L(\tau_L(b), \underline{\omega}) - \underline{\omega} g_L(b) + b_L'(b) - (1+\rho)b > R_H(\tau_H(b), \underline{\omega}) - \underline{\omega} g_H(b) + b_H'(b) - (1+\rho)b = 0.$$

This implies $\lambda_L(b) = 0$. So we have $\lambda_L(b) < \lambda_H(b)$, a contradiction. ■

We now prove:

Lemma A.9. *In equilibrium: (i) $b_H'(b) > b$ for all $b \in [\underline{b}, \overline{b})$, and, (ii) $b_L'(b) > b$ for all $b \in [\underline{b}, b^q)$ and $b_L'(b) < b$ for all $b \in (b^q, \overline{b}]$.*

Proof (i) We need to show that $b_H'(b) > b$ for all $b \in [\underline{b}, \overline{b})$. Let $b \in [\underline{b}, \overline{b})$. Suppose first that $b \leq b_H^q$. Then, we have that $b_H'(b) = b^q \geq b_H^q > b$. Suppose next that $b > b_H^q$. We know that $b_H'(b) > b^q$ and that $b_H'(b)$ satisfies the first order condition:

$$1 + \lambda_H(b) = -\beta E V_\theta'(b_H'(b))$$

where $\lambda_H(b)$ is the Lagrangian multiplier on the budget constraint on the maximization problem (53). We also know from the proof of Lemma A.6 that

$$-\beta V_\theta'(b) = \begin{cases} 1 + \lambda_\theta(b) & \text{if } b > b_\theta^q \\ 1 & \text{if } b < b_\theta^q \end{cases} \tag{66}$$

Suppose that $b_H'(b) \leq b$. Then if $b \geq b_L^q$, we have that

$$1 + \lambda_H(b) = -\beta E V_\theta'(b_H'(b)) \leq -\beta E V_\theta'(b) = \alpha(1 + \lambda_H(b)) + (1-\alpha)(1 + \lambda_L(b)) < 1 + \lambda_H(b)$$

since $\lambda_H(b) > \lambda_L(b)$ for all $b \geq b_H^q$ by Lemma A.8. If $b < b_L^q$, we have that

$$1 + \lambda_H(b) = -\beta E V_\theta'(b_H'(b)) \leq -\beta E V_\theta'(b) = \alpha(1 + \lambda_H(b)) + (1-\alpha) < 1 + \lambda_H(b).$$

(ii) We first show that $b_L'(b) > b$ for all $b \in [\underline{b}, b^q)$. Let $b \in [\underline{b}, b^q)$. Then since $b^q < b_L^q$, we know that $b_L'(b) = b^q > b$. We next show that $b_L'(b) < b$ for all $b \in (b^q, \overline{b}]$. Let $b \in (b^q, \overline{b}]$. Suppose first that $b \leq b_L^q$. Then we know that $b_L'(b) = b^q < b$. Now suppose that $b > b_L^q$. We know that $b_L'(b)$ satisfies the first order condition:

$$1 + \lambda_L(b) = -\beta E V_\theta'(b_L'(b)).$$

Suppose that $b'_L(b) \geq b$. Then since $b > b^q_L$ we have that

$$1 + \lambda_L(b) = -\beta EV'_\theta(b'_L(b)) \geq -\beta EV'_\theta(b) = \alpha(1 + \lambda_H(b)) + (1 - \alpha)(1 + \lambda_L(b)) > 1 + \lambda_L(b),$$

where the last step relies on (66) and the fact that by Lemma A.8 $\lambda_H(b) > \lambda_L(b)$. This is a contradiction. ■

6.5.3 The stable distribution

Let $\psi_t(b)$ denote the distribution function of the current level of debt at the beginning of period t. The distribution function $\psi_0(b)$ is exogenous and is determined by the economy's initial level of debt b_0. The transition function implied by the equilibrium is given by

$$H(b, b') = \begin{cases} \Pr\{\theta' \text{ s.t. } b'_{\theta'}(b) \leq b'\} & \text{if } \exists\, \theta' \;\; s.t.\; b'_{\theta'}(b) \leq b' \\ 0 & otherwise \end{cases},$$

for any $b' \in [b^q, \overline{b}]$. $H(b, b')$ is the probability that in the next period the initial level of debt will be less than or equal to $b' \in [b^q, \overline{b}]$ if the current level of debt is b. Using this notation, the distribution of debt at the beginning of any period $t \geq 1$ is defined inductively by $\psi_t(b) = \int_z H(z, b) d\psi_{t-1}(z)$. The sequence of distributions $\langle \psi_t(b) \rangle$ converges to the distribution $\psi(b)$ if we have that $\lim_{t\to\infty} \psi_t(b) = \psi(b)$ for all $b' \in [b^q, \overline{b}]$. Moreover, $\psi^*(b)$ is an invariant distribution if

$$\psi^*(b) = \int_z H(z, b) d\psi^*(z).$$

We now establish that any sequence of equilibrium debt distributions $\langle \psi_t(b) \rangle$ converges to a unique invariant distribution $\psi^*(b)$.

It is easy to prove that the transition function $H(b, b')$ has the Feller Property and that it is monotonic in b (see Ch. 8.1 in Stokey, Lucas and Prescott (1989) for definitions). Define the function $H^m(b, b')$ inductively by $H^0(b, b') = H(b, b')$ and $H^m(b, b') = \int_z H(z, b') dH^{m-1}(b, z)$. By Theorem 12.12 in Stokey, Lucas and Prescott (1989), therefore, the result follows if the following "mixing condition" is satisfied:

Mixing Condition: *There exists an* $\epsilon > 0$ *and* $m \geq 0$*, such that* $H^m(\overline{b}, b^q) \geq \epsilon$ *and* $H^m(\underline{b}, b^q) \leq 1 - \epsilon$*.*

We proceed in two steps.

Step 1. We first show that there exists an $\epsilon > 0$ and $m \geq 0$, such that $H^m(\overline{b}, b^q) \geq \epsilon$. Assume by contradiction that $H^m(\overline{b}, b_L^q) = 0$ for any m. Then the political equilibrium coincides with the planner's solution, since with probability one: $B_\theta(\tau_\theta'(b), g_\theta'(b), b_\theta'(b), b, \underline{\omega}) = 0$. By the argument of Proposition 2, this implies: $\Pr\left(\lim_{n\to\infty} \lambda_{\theta_n}(b_n) = 0\right) = 1$. So $\Pr\left(\lim_{n\to\infty} b'_{\theta_n}(b_n) > \underline{b}\right) = 0$, but then $\Pr\left(\lim_{n\to\infty} b'_{\theta_n}(b_n) > b_L^q\right) = 0$, a contradiction. So there must be an $\varepsilon > 0$ and $m \geq 0$, such that $H^{m-1}(\overline{b}, b_L^q) > \varepsilon$. This implies $H^m(\overline{b}, b^q) \geq (1-\alpha)\, H^{m-1}(\overline{b}, b_L^q) > 0$.

Step 2. We now show that there exists an $\varepsilon > 0$ and $m \geq 0$, such that $1 - H^m(\underline{b}, b^q) \geq \varepsilon$. With probability $H^{m-1}(\underline{b}, b_L^q)$ the level of debt chosen in period $m-1$ is b_L^q when the initial level of debt is $\underline{b}$. Given this, the probability that the level of debt is larger than b^q in period m is at least $H^{m-1}(\underline{b}, b_L^q)\,[1 - H(b_L^q, b^q)]$. By the previous step and the monotonicity of $H(b, b')$ we have that there is a $\varepsilon > 0$ such that $H^{m-1}(\underline{b}, b_L^q) > \varepsilon$. Since $b_L^q > b_H^q$, we have $b'_H\left(b_L^q\right) \geq b^q$: it follows that $[1 - H(b_L^q, b^q)]\, H^{m-1}(\underline{b}, b_L^q) \geq \alpha\varepsilon > 0$.

To prove that the stable distribution has full support in $[b^q, \overline{b})$ we now show that for any $b \in [b^q, \overline{b})$, $\psi^*(b) \in (0,1)$. The fact that $\psi^*(b) > 0$, follows from Step 1 presented above. We therefore only need to show that $\psi^*(b) < 1$. Let $b_0 \in \left[b_H^q, \overline{b}\right]$. Define recursively a sequence b_t such that $b_t = b_H(b_{t-1})$. This sequence is monotonically increasing and bounded, so it converges. Assume that $\lim_{t\to\infty} b_t = b_\infty < \overline{b}$. We have that $b'_H(b_t) \leq b_t + \varepsilon_t$, where $\varepsilon_t > 0$ and $\varepsilon_t \to 0$ as $t \to \infty$. We therefore have:

$$\begin{aligned} 1 + \lambda_H(b_t) &\leq -\beta E V_\theta'(b_t + \varepsilon_t) \\ &= \alpha(1 + \lambda_H(b_t + \varepsilon_t)) + (1-\alpha)(1 + \lambda_L(b_t + \varepsilon_t)) \\ &< 1 + \lambda_H(b_t + \varepsilon_t) - (1-\alpha)\Delta^*, \end{aligned} \tag{67}$$

where the last inequality follows from the fact that by Lemma A.8 $\lambda_H(b) > \lambda_L(b)$ for all $b \in [b^q, b_\infty]$, so there is a $\Delta^* > 0$ so that $\lambda_H(b) - \Delta^* > \lambda_L(b)$ in a left neighborhood of b_∞. But since $\lambda_L(b_t)$ is continuous in b, (67) implies $\lambda_H(b_\infty) < \lambda_H(b_\infty) - (1-\alpha)\Delta^*$, a contradiction.

The argument above implies that for any $b < \overline{b}$, there is a finite T such that starting from any $b_0 \geq b^q$, we have $b_T > b$ with strictly positive probability. This implies that $\psi^*(b) < 1$. ■

6.6 Proof of Proposition 7

We begin by proving the two properties stated in the text before Proposition 7. These are (i) that $r_\theta(b)$ is increasing in b for each state θ, and (ii) that $r_H(b^q) > r_H^q$, and $r_L(b^q) \leq r_L^q$. To see the first property, assume first that $b \leq b^q_\theta$. In this case $b'_\theta(b) = b^q$, so $r_\theta(b) = (1+\rho)\, b - b'_\theta(b)$ is increasing in b. Assume now that $b > b^q_\theta$. Then we know that $B_\theta(\tau_\theta(b), g_\theta(b), b'_\theta(b), b, \underline{\omega}) = 0$, implying that $b(1+\rho) - b'_\theta(b) = R_\theta(\tau_\theta(b), \underline{\omega}) - \underline{\omega} g_\theta(b)$. An increase in b implies that $\lambda_\theta(b)$ increases, implying that $R_\theta(\tau_\theta(b), \underline{\omega}) - \underline{\omega} g_\theta(b)$ increases in b. The fact that $r_H(b^q) \geq r^q_H$, and $r_L(b^q) \leq r^q_L$ follows from Lemma A.7.

We can now prove the Proposition. We consider only the case in which $q > q^*_L$, since the arguments for the other two cases are similar. Since $q^*_L > q^*_H$, we know that $q > q^*_H$. From the two properties we have just established, we know that $r_H(b) \geq r^q_H$ for all $b \in [b^q, \overline{b})$. It follows from Proposition 5 that there is unemployment in the high cost state. It follows from Proposition 5 and the definition of b^q_L that in the low cost state, there is unemployment with optimal revenue extraction if $b \in [b^q, b^q_L]$ and unemployment if $b > b^q_L$. That unemployment is increasing in b in the high cost state and in the low cost state when $b > b^q_L$ follows from Proposition 5, the definition of b^q_L, and the fact that the equilibrium revenue requirement is increasing in b.

It only remains to show that for given b, unemployment is greater in the high than the low cost state. Suppose first that $b \in [b^q, b^q_L]$. Then employment in the low cost state is $g^q_L + x_L(\tau^q_L)/A$ where (τ^q_L, g^q_L) is given by (30). Substituting in for (τ^q_L, g^q_L) we find that employment in the low cost state is

$$n_e \left[\frac{q(A_L - \underline{w}) + \underline{w}}{\xi(2q-1)} \right] + \frac{\gamma}{(q-1)\, \underline{w}}. \tag{68}$$

In the high cost state, employment is strictly less than $g^q_H + x_H(\tau^q_H)/A$ where (τ^q_H, g^q_H) is given by (30). Thus, employment in the high cost state is smaller than

$$n_e \left[\frac{q(A_H - \underline{w}) + \underline{w}}{\xi(2q-1)} \right] + \frac{\gamma}{(q-1)\, \underline{w}}. \tag{69}$$

It is now easy to see that (69) is smaller than (68), which implies the result.

Now suppose that $b > b^q_L$. Then in each state θ, $(\tau_\theta(b), g_\theta(b))$ satisfies the first order conditions

$$\frac{\gamma}{g} = \lambda_\theta(b)\underline{\omega}, \tag{70}$$

and

$$\lambda_\theta(b)(1-2\tau)(A_\theta - \underline{\omega}) = \tau A_\theta + (1-\tau)\underline{\omega}. \tag{71}$$

Moreover, by Lemma A.8, we know that $\lambda_H(b) > \lambda_L(b)$. It is clear that (70) implies that $g_H(b) < g_L(b)$. Thus, to prove the claim, we just need to show that $x_H(\tau_H(b)) < x_L(\tau_L(b))$. From (71), we have that

$$\tau_\theta(b) = \frac{\lambda_\theta(b)(A_\theta - \underline{\omega}) - \underline{\omega}}{(A_\theta - \underline{\omega})(1 + 2\lambda_\theta(b))}.$$

From (8), this implies that

$$x_\theta(\tau_\theta(b)) = \frac{n_e A\left[A_\theta + \lambda_\theta(b)(A_\theta - \underline{\omega})\right]}{\xi(1 + 2\lambda_\theta(b))}.$$

Note that the expression on the right hand side is increasing in A_θ and decreasing in $\lambda_\theta(b)$. Thus, we have that

$$x_L(\tau_L(b)) > \frac{n_e A\left[A_L + \lambda_H(b)(A_L - \underline{\omega})\right]}{\xi(1 + 2\lambda_H(b))} > \frac{n_e A\left[A_H + \lambda_H(b)(A_H - \underline{\omega})\right]}{\xi(1 + 2\lambda_H(b))} = x_H(\tau_H(b)),$$

as required. ■