Subject : MATHEMATICS

Paper 1 : ABSTRACT ALGEBRA

Chapter 1 : Direct Product of Groups

Module 1 : External direct product of groups

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External direct product of groups

Learning outcomes:	1. External direct product of groups.
	2. Product of cyclic groups.
	3. Order of an element in the product of two groups.
	4. Quotient of direct products.

In this module we define a suitable binary operation $*$ on the cartesian product $G_1 \times G_2$ of two groups G_1 and G_2 so that $(G_1 \times G_2, *)$ is a group which we call the external direct product of groups. This is a widely used technique to construct a new group from old as well as to decompose a group as a product of relatively better known groups. Thus external direct product of groups is an important notion in the structure theory of finite groups.

Let $G_1, G_2, \cdots, G_n$ be n groups. Define a binary operation $*$ on the cartesian product $G = G_1 \times G_2 \times \cdots \times G_n$ by:

$$(a_1, a_2, \cdots, a_n) * (b_1, b_2, \cdots, b_n) = (a_1b_1, a_2b_2, \cdots, a_nb_n) \tag{0.1}$$

for every $(a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) \in G$. Here a_ib_i is the product of two elements a_i and b_i in G_i for every $i = 1, 2, \cdots, n$.

For example, the binary operation $*$ on $K_4 \times \mathbb{Z}$ is given by:

$$(a, m) * (b, n) = (ab, m + n), \text{ for every } (a, m), (b, n) \in K_4 \times \mathbb{Z}.$$

It is easy to check that $(K_4 \times \mathbb{Z}, *)$ is a group having the identity element $(e, 0)$ and for every $(a, n) \in K_4 \times \mathbb{Z}$, $(a, n)^{-1} = (a, -n)$.

Similarly $(G, *)$ is a group with the identity element $e = (e_1, e_2, \cdots, e_n)$ and $(a_1, a_2, \cdots, a_n)^{-1} = (a_1^{-1}, a_2^{-1}, \cdots, a_n^{-1})$ for every $(a_1, a_2, \cdots, a_n) \in G$ where e_i is the identity element of G_i for every $i = 1, 2, \cdots, n$.

Definition 0.1. *Let $G_1, G_2, \cdots, G_n$ be n groups and $G = G_1 \times G_2 \times \cdots \times G_n$. Then the group $(G, *)$ is called the external direct product of the groups $G_1, G_2, \cdots, G_n$.*

Example 0.2. *Consider the noncommutative group $(S_3, \cdot)$ and the infinite group $(\mathbb{Z}, +)$. Then the binary operation $*$ on $G = S_3 \times \mathbb{Z}$ is given by:*

$$(\alpha, m) * (\beta, n) = (\alpha\beta, m + n).$$

The identity element in G is $(e, 0)$ and $(\alpha, m)^{-1} = (\alpha^{-1}, -m)$ for every $\alpha \in S_3$ and $m \in \mathbb{Z}$. For example,

$$((12), 4)^{-1} = ((12), -4)$$
$$\text{and } ((123), 2)^{-1} = ((132), -2).$$

Now,

$$((12), 4) * ((13), 7) = ((12)(13), 4 + 7) = ((132), 11)$$
$$\text{and } ((13), 7) * ((12), 4) = ((13)(12), 7 + 4) = ((123), 11)$$

shows that the group $S_3 \times \mathbb{Z}$ is noncommutative. Also $S_3 \times \mathbb{Z}$ is an infinite group. What makes it interesting is that noncommutativity of $S_3 \times \mathbb{Z}$ comes from S_3, whereas infiniteness of $\mathbb{Z}$ makes $S_3 \times \mathbb{Z}$ an infinite group.

Theorem 0.3. *Let G_1 and G_2 be two groups. Then $G_1 \times G_2 \simeq G_2 \times G_1$.*

Proof. Define $f : G_1 \times G_2 \longrightarrow G_2 \times G_1$ by:

$$f(a, b) = (b, a) \text{ for all } (a, b) \in G_1 \times G_2.$$

Then

$$\begin{aligned} f((a_1, b_1) * (a_2, b_2)) &= f(a_1 a_2, b_1 b_2) \\ &= (b_1 b_2, a_1 a_2) \\ &= (b_1, a_1) * (b_2, a_2) \\ &= f(a_1, b_1) * f(a_2, b_2). \end{aligned}$$

shows that f is a homomorphism. Also f is one-to-one and onto. Thus f is an isomorphism, and hence $G_1 \times G_2 \simeq G_2 \times G_1$. □

Theorem 0.4. *Let $G_1, G_2, \cdots, G_n$ be n-groups. Then the group $G = G_1 \times G_2 \times \cdots \times G_n$ is abelian if and only if each of the groups $G_1, G_2, \cdots, G_n$ is abelian.*

Proof. First assume that G is abelian. Consider $1 \leq i \leq n$ and $a, b \in G_i$. Then $(e_1, \cdots, e_{i-1}, a, e_{i+1}, \cdots, e_n), (e$
G and commutativity of G implies that

$$\begin{aligned} &(e_1, \cdots, e_{i-1}, a, e_{i+1}, \cdots, e_n) * (e_1, \cdots, e_{i-1}, b, e_{i+1}, \cdots, e_n) \\ &\qquad = (e_1, \cdots, e_{i-1}, b, e_{i+1}, \cdots, e_n) * (e_1, \cdots, e_{i-1}, a, e_{i+1}, \cdots, e_n) \\ &\Rightarrow (e_1, \cdots, e_{i-1}, ab, e_{i+1}, \cdots, e_n) = (e_1, \cdots, e_{i-1}, ba, e_{i+1}, \cdots, e_n) \\ &\Rightarrow ab = ba. \end{aligned}$$

Thus G_i is abelian for every $1 \leq i \leq n$.

Conversely, suppose that G_i is abelian for every $1 \leq i \leq n$. Consider $(a_1, a_2, \cdots, a_n), (b_1, b_2, \cdots, b_n) \in G$. Then $a_i, b_i \in G_i$, and since G_i is abelian, so $a_i b_i = b_i a_i$ for every $1 \leq i \leq n$. Now

$$\begin{aligned}(a_1, a_2, \cdots, a_n) * (b_1, b_2, \cdots, b_n) &= (a_1 b_1, a_2 b_2, \cdots, a_n b_n)\\ &= (b_1 a_1, b_2 a_2, \cdots, b_n a_n)\\ &= (b_1, b_2, \cdots, b_n) * (a_1, a_2, \cdots, a_n)\end{aligned}$$

implies that G is abelian. □

Following generalization of this theorem shows that even if G_1 and G_2 are not abelian then also the extent of commutativity can be measured by that of G_1 and G_2.

Theorem 0.5. *Let $G_1, G_2, \cdots, G_n$ be n-groups. Then $Z(G_1 \times G_2 \times \cdots \times G_n) \simeq Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$.*

Proof. This is sufficient to prove the result for $n = 2$. Let G_1 and G_2 be two groups. Consider $a \in G_1$ and $b \in G_2$. Then we have,

$$\begin{aligned}&(a, b) \in Z(G_1 \times G_2)\\ \Leftrightarrow\ &(a, b) * (g_1, g_2) = (g_1, g_2) * (a, b) \text{ for all } (g_1, g_2) \in G_1 \times G_2\\ \Leftrightarrow\ &(ag_1, bg_2) = (g_1 a, g_2 b) \text{ for all } (g_1, g_2) \in G_1 \times G_2\\ \Leftrightarrow\ &ag_1 = g_1 a \text{ and } bg_2 = g_2 b \text{ for all } g_1 \in G_1, g_2 \in G_2\\ \Leftrightarrow\ &a \in Z(G_1) \text{ and } b \in Z(G_2)\\ \Leftrightarrow\ &(a, b) \in Z(G_1) \times Z(G_2).\end{aligned}$$

Thus $Z(G_1 \times G_2) \simeq Z(G_1) \times Z(G_2)$. □

Now we show that the product of two cyclic groups may not be cyclic. For example, consider the external direct product $\mathbb{Z}_4 \times \mathbb{Z}_6$ of the cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$. Then $\mathbb{Z}_4 \times \mathbb{Z}_6$ contains 24 elements, but, for every $([a]_4, [b]_6) \in \mathbb{Z}_4 \times \mathbb{Z}_6$, $12([a]_4, [b]_6) = ([12a]_4, [12b]_6) = ([0]_4, [0]_6)$ implies that $\mathbb{Z}_4 \times \mathbb{Z}_6$ has no element of order 24. Therefore $\mathbb{Z}_4 \times \mathbb{Z}_6$ is not a cyclic group. Also this example gives us a hope that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ may be cyclic if $lcm\ (m, n) = mn$, equivalently, if m and n are relatively prime.

Now we prove a necessary and sufficient condition for the external direct product of two finite cyclic groups to be cyclic.

Theorem 0.6. *Let G_1 and G_2 be two finite cyclic groups of order m and n respectively. Then $G_1 \times G_2$ is cyclic if and only if $\gcd(m, n) = 1$.*

In this case, (a, b) is a generator of $G_1 \times G_2$ if and only if a and b are generators of G_1 and G_2 respectively.

Proof. First assume that $G_1 \times G_2$ is a cyclic group and $G_1 \times G_2 =< (a, b) >$. Then $o((a, b)) = mn$. Let $d = \gcd(m, n)$. Now $(a, b)^{\frac{mn}{d}} = (a^{m\frac{n}{d}}, b^{n\frac{m}{d}}) = (e_1, e_2)$ implies that $mn \mid \frac{mn}{d}$. Hence it follows that $d = 1$.

Conversely, let $\gcd(m, n) = 1$. Suppose that $G_1 =< a >$ and $G_2 =< b >$. Denote $k = o(a, b)$. Then $(a, b)^{mn} = (a^{mn}, b^{mn}) = (e_1, e_2)$ implies that $k \mid mn$. Now $(a^k, b^k) = (a, b)^k = (e_1, e_2)$ implies that $a^k = e_1$ and $b^k = e_2$; and so $m \mid k$ and $n \mid k$. Then $\gcd(m, n) = 1$ implies that $mn \mid k$. Therefore $k = mn = |G_1 \times G_2|$. Hence $G_1 \times G_2$ is a cyclic group and (a, b) is a generator. □

The following consequence is immediate.

Corollary 0.7. $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ *if and only if* $\gcd(m, n) = 1$.

Thus we see that for $a \in G_1$ of order m and $b \in G_2$ of order n, if $\gcd(m, n) = 1$, then $o((a, b)) = mn$. What does it happen if $\gcd(m, n) \neq 1$?

Theorem 0.8. *Let G_1 and G_2 be two finite groups. Then for every $a \in G_1, b \in G_2$, the order of (a, b) in $G_1 \times G_2$ is $o((a, b)) = lcm(o(a), o(b))$.*

Proof. Since G_1 and G_2 are finite groups, so $G_1 \times G_2$ is finite. Thus for every $a \in G_1, b \in G_2$, $o(a)$ in G_1, $o(b)$ in G_2 and $o((a, b))$ in $G_1 \times G_2$ all are finite. Now we have,

$$\begin{aligned} & o(a, b) \text{ is the smallest positive integer } n \text{ such that } (a, b)^n = (e_1, e_2), \\ \Leftrightarrow\ & n \text{ is the smallest positive integer such that } a^n = e_1 \text{ and } b^n = e_2, \\ \Leftrightarrow\ & n \text{ is the smallest positive integer such that } o(a) \mid n \text{ and } o(b) \mid n. \end{aligned}$$

Thus $o(a, b) = lcm\ (o(a), o(b))$. □

External direct product of two infinite cyclic groups is not cyclic. Since every infinite cyclic group is isomorphic to $\mathbb{Z}$, it is sufficient to consider $\mathbb{Z} \times \mathbb{Z}$. On the contrary, if possible, let $\mathbb{Z} \times \mathbb{Z}$ be cyclic and $\mathbb{Z} \times \mathbb{Z} =< (a, b) >$. Since $(1, 1) \in \mathbb{Z} \times \mathbb{Z}$, so there is $r \in \mathbb{Z}$ such that $r(a, b) = (1, 1)$ which implies that $a = b = 1$ or -1. Then $< (a, b) >= \{(n, n) \mid n \in \mathbb{Z}\} \neq \mathbb{Z} \times \mathbb{Z}$. This contradiction shows that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.

We conclude this module with the characterization of quotients of $G_1 \times G_2$.

Theorem 0.9. *Let G_1 and G_2 be two groups. Then*

$$G_1 \times G_2 / G_1 \times \{e_2\} \simeq G_2 \text{ and } G_1 \times G_2 / \{e_1\} \times G_2 \simeq G_1.$$

Proof. Define $f : G_1 \times G_2 \longrightarrow G_2$ by:

$$f(a, b) = b \text{ for all } (a, b) \in H \times K.$$

Then f is a homomorphism and

$$\begin{aligned}\ker f &= \{(a,b) \in G_1 \times G_2 \mid f(a,b) = e_2\} \\ &= \{(a,b) \in G_1 \times G_2 \mid b = e_2\} \\ &= \{(a,e_2) \mid a \in G_1\} \\ &= G_1 \times \{e_2\}.\end{aligned}$$

Also f is onto. Hence, by the First Isomorphism Theorem, $G_1 \times G_2/G_1 \times \{e_2\} \simeq G_2$. Similarly, we have $G_1 \times G_2/\{e_1\} \times G_2 \simeq G_1$. □

Also we have:

Theorem 0.10. *Let G_1 and G_2 be two groups. If N_1 and N_2 are two normal subgroups of G_1 and G_2 respectively, then $N_1 \times N_2$ is a normal subgroup of $G_1 \times G_2$, and*

$$G_1 \times G_2/N_1 \times N_2 \simeq G_1/N_1 \times G_2/N_2.$$

Proof. Define $f : G_1 \times G_2 \longrightarrow G_1/N_1 \times G_2/N_2$ by:

$$f(a,b) = (aN_1, bN_2) \qquad \text{for every } (a,b) \in G_1 \times G_2.$$

Then for every $(a,b),(c,d) \in G_1 \times G_2$,

$$\begin{aligned}f((a,b) * (c,d)) &= f(ac, bd) \\ &= (acN_1, bdN_2) \\ &= (aN_1cN_1, bN_2dN_2) \\ &= (aN_1, bN_2) * (cN_1, dN_2) \\ &= f(a,b) * f(c,d)\end{aligned}$$

implies that f is a homomorphism. Also f is onto. Then it follows from the First Isomorphism Theorem that

$$G_1 \times G_2/\ker f \simeq G_1/N_1 \times G_2/N_2.$$

Now we have,

$$\begin{aligned}\ker f &= \{(a,b) \in G_1 \times G_2 \mid f(a,b) = (N_1, N_2)\} \\ &= \{(a,b) \in G_1 \times G_2 \mid (aN_1, bN_2) = (N_1, N_2)\} \\ &= \{(a,b) \in G_1 \times G_2 \mid aN_1 = N_1 \text{ and } bN_2 = N_2\} \\ &= \{(a,b) \in G_1 \times G_2 \mid a \in N_1 \text{ and } b \in N_2\} \\ &= N_1 \times N_2.\end{aligned}$$

Thus the result follows. □

1 Summary

- Let $G_1, G_2, \cdots, G_n$ be n groups and $G = G_1 \times G_2 \times \cdots \times G_n$. Then the binary operation $*$ on G defined by:

$$(a_1, a_2, \cdots, a_n) * (b_1, b_2, \cdots, b_n) = (a_1b_1, a_2b_2, \cdots, a_nb_n)$$

  is such that $(G, *)$ is a group. The group $(G, *)$ is called the external direct product of the groups $G_1, G_2, \cdots, G_n$.

- The group $S_3 \times \mathbb{Z}$ is infinite and noncommutative.

- Let G_1 and G_2 be two groups. Then $G_1 \times G_2 \simeq G_2 \times G_1$.

- Let G_1 and G_2 be two groups. Then the group $G = G_1 \times G_2$ is abelian if and only if both G_1 and G_2 are abelian.

- Let G_1 and G_2 be two groups. Then $Z(G_1 \times G_2) \simeq Z(G_1) \times Z(G_2)$.

- The external direct product $\mathbb{Z}_4 \times \mathbb{Z}_6$ of the cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$ is not a cyclic group.

- Let G_1 and G_2 be two finite cyclic groups of order m and n respectively. Then $G_1 \times G_2$ is cyclic if and only if $\gcd(m, n) = 1$.

  In this case, (a, b) is a generator of $G_1 \times G_2$ if and only if a and b are generators of G_1 and G_2 respectively.

- $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.

- Let G_1 and G_2 be two finite groups. Then for every $a \in G_1, b \in G_2$, the order of (a, b) in $G_1 \times G_2$ is $o((a, b)) = lcm(o(a), o(b))$.

- External direct product of two infinite cyclic groups is not cyclic.

- Let G_1 and G_2 be two groups. Then

$$G_1 \times G_2/G_1 \times \{e_2\} \simeq G_2 \text{ and } G_1 \times G_2/\{e_1\} \times G_2 \simeq G_1.$$

- Let G_1 and G_2 be two groups. If N_1 and N_2 are two normal subgroups of G_1 and G_2 respectively, then $N_1 \times N_2$ is a normal subgroup of $G_1 \times G_2$, and

$$G_1 \times G_2/N_1 \times N_2 \simeq G_1/N_1 \times G_2/N_2.$$