Cauchy Riemann in polar coordinates.

Suppose f is a complex valued function that is differentiable at a point z_0 of the complex plane. The idea here is to modify the method that resulted in the "cartesian" version of the Cauchy-Riemann equations derived in §17 to get the polar version.

To this end, suppose $z_0 \neq 0$, write $z = re^{i\theta}$, $z_0 = r_0 e^{i\theta_0}$ and express the real and imaginary parts of f as functions of r and θ:

$$f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta).$$

STEP I. In the definition of "differentiable at z_0," let $z \to z_0$ along the ray $\theta = \theta_0$ (draw a picture to illustrate this!). Then the following limit exists:

$$\begin{aligned} f'(z_0) &= \lim_{r\to r_0} \frac{f(re^{i\theta_0}) - f(r_0e^{i\theta_0})}{re^{i\theta_0} - r_0e^{i\theta_0}} \\ &= \frac{1}{e^{i\theta_0}} \lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0) + i\,[v(r,\theta_0) - v(r_0,\theta_0)]}{r - r_0} \\ &= e^{-i\theta_0}\left[\lim_{r\to r_0} \frac{u(r,\theta_0) - u(r_0,\theta_0)}{r - r_0} + i \lim_{r\to r_0} \frac{v(r,\theta_0) - v(r_0,\theta_0)}{r - r_0}\right]. \end{aligned}$$

Both limits in the last line exist because the limit in the first line does (and a complex function has a limit at a point if and only if its real and imaginary parts do). Now these limits equal the respective partial derivatives of u and v with respect to r, at the polar coordinates (r_0, θ_0). The result is:

$$f'(z_0) = e^{-i\theta_0}\left[\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right] \tag{1}$$

STEP II. In the definition of "differentiable at z_0," let $z \to z_0$ along the circle $r = r_0$ (draw another picture to illustrate this new situation!), so that the following is true:

$$\begin{aligned} f'(z_0) &= \lim_{\theta\to\theta_0} \frac{f(r_0e^{i\theta}) - f(r_0e^{i\theta_0})}{r_0e^{i\theta} - r_0e^{i\theta_0}} \\ &= \frac{1}{r_0} \lim_{\theta\to\theta_0} \left[\frac{u(r_0,\theta) - u(r_0,\theta_0)}{e^{i\theta} - e^{i\theta_0}} + i\,\frac{v(r_0,\theta) - v(r_0,\theta_0)}{e^{i\theta} - e^{i\theta_0}}\right] \\ &= \frac{1}{r_0} \lim_{\theta\to\theta_0} \left\{\left[\frac{u(r_0,\theta) - u(r_0,\theta_0)}{\theta - \theta_0} + i\,\frac{v(r_0,\theta) - v(r_0,\theta_0)}{\theta - \theta_0}\right] \times \frac{\theta - \theta_0}{e^{i\theta} - e^{i\theta_0}}\right\}. \end{aligned}$$

As $\theta \to \theta_0$ the difference quotients in the square brackets converge—if they converge at all—to the partial derivatives of u and v with respect to θ, evaluated at the polar coordinates (r_0, θ_0). This much-desired convergence will happen if we can prove that the last fraction has a limit as $\theta \to \theta_0$ (make sure you understand why this is true!). Now the reciprocal of the fraction whose convergence we hope to establish is:

$$\frac{e^{i\theta} - e^{i\theta_0}}{\theta - \theta_0} = \frac{\cos\theta - \cos\theta_0}{\theta - \theta_0} + i\,\frac{\sin\theta - \sin\theta_0}{\theta - \theta_0}\ ,$$

which, as $\theta \to \theta_0$, tends to

$$\left[\frac{d}{d\theta}\cos\theta\right]_{\theta=\theta_0} + i\left[\frac{d}{d\theta}\sin\theta\right]_{\theta=\theta_0} = -\sin\theta_0 + i\cos\theta_0 = ie^{i\theta_0}.$$

Putting it all together:

$$f'(z_0) = \frac{1}{i\,r_0 e^{i\theta_0}}\left[\frac{\partial u}{\partial\theta}(r_0,\theta_0) + i\,\frac{\partial v}{\partial\theta}(r_0,\theta_0)\right]$$

which, after a little complex arithmetic, becomes:

$$f'(z_0) = \frac{e^{-i\theta_0}}{r_0}\left[\frac{\partial v}{\partial\theta}(r_0,\theta_0) - i\,\frac{\partial u}{\partial\theta}(r_0,\theta_0)\right] \tag{2}$$

STEP III. Equations (1) and (2) give two expressions for $f'(z_0)$. Upon equating the real and imaginary parts of the right-hand sides of these equations we arrive at the *Polar - auchy-Riemann Equations-*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta} \tag{3}$$

where, for aesthetic reasons, I've left as understood the fact that everything is supposed to be evaluated at the polar coordinates (r_0, θ_0).

SUMMARY. *If $f = u + i\,v$ is differentiable at $z_0 = r_0 e^{i\theta_0} \neq 0$, then the polar Cauchy-Riemann equations (3) hold at (r_0, θ_0). In addition, we have these formulas for the derivative of f:*

$$f'(z_0) = e^{-i\theta_0}\left[\frac{\partial u}{\partial r}(r_0,\theta_0) + i\frac{\partial v}{\partial r}(r_0,\theta_0)\right] = \frac{e^{-i\theta_0}}{r_0}\left[\frac{\partial v}{\partial\theta}(r_0,\theta_0) - i\,\frac{\partial u}{\partial\theta}(r_0,\theta_0)\right]\ .$$